z Transform

Chapter Intended Learning Outcomes:

(i) Represent discrete-time signals using z transform

(ii) Understand the relationship between z transform and discrete-time Fourier transform

(iii) Understand the properties of z transform

(iv) Perform operations on z transform and inverse z transform

(v) Apply z transform for analyzing linear time-invariant systems

Discrete-Time Signal Representation with z Transform

Apart from discrete-time Fourier transform (DTFT), we can also use z transform to represent discrete-time signals.

The z transform of $x[n]$, denoted by $X(z)$, is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \quad (8.1)$$

where z is a continuous complex variable.

We can also express z as:

$$z = re^{j\omega} \quad (8.2)$$

where $r = |z| > 0$ is magnitude and $\omega = \angle(z)$ is angle of z.

Employing (8.2), the z transform can be written as:

$$X(z)|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} \quad (8.3)$$

Comparing (8.3) and the DTFT formula in (6.4):

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (8.4)$$

That is, z transform of $x[n]$ is equal to the DTFT of $x[n]r^{-n}$.

When $r = 1$ or $z = e^{j\omega}$, (8.3) and (8.4) are identical:

$$X(z)|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (8.5)$$

That is, z transform generalizes the DTFT.

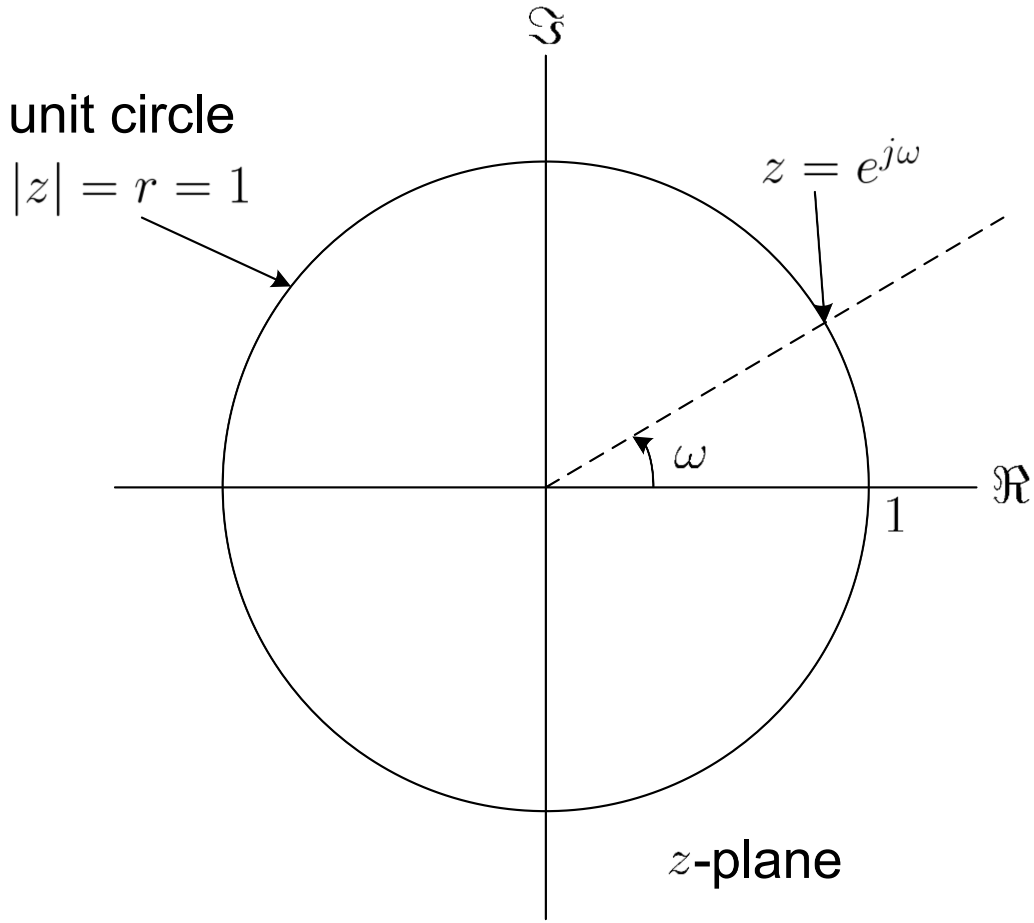

Fig.8.1: Relationship between $X(z)$ and $X(e^{j\omega})$ on the z-plane

Region of Convergence (ROC)

ROC indicates when z transform of a sequence converges.

Generally there exists some z such that

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \to \infty \quad (8.6)$$

where the z transform does not converge.

The set of values of z for which $X(z)$ converges or

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n] z^{-n} \right| \le \sum_{n=-\infty}^{\infty} \left| x[n] z^{-n} \right| < \infty \quad (8.7)$$

is called the ROC, which must be specified along with $X(z)$ in order for the z transform to be complete.

Note also that if

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \rightarrow \infty \quad (8.8)$$

then the DTFT does not exist. While the DTFT converges if

$$\left|X(e^{j\omega})\right| = \left|\sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}\right| \leq \sum_{n=-\infty}^{\infty} \left|x[n]e^{-j\omega n}\right| = \sum_{n=-\infty}^{\infty} |x[n]| < \infty \quad (8.9)$$

Comparing (8.7) and (8.9), we see that even $x[n]$ is not absolutely summable, (8.7) may hold for certain z.

On the other hand, the z transform does not exist if there is no value of z satisfies (8.7).

Assuming that $x[n]$ is of infinite length, we decompose $X(z)$:

$$X(z) = X_-(z) + X_+(z) \tag{8.10}$$

where

$$X_-(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n} = \sum_{m=1}^{\infty} x[-m]z^{m} \tag{8.11}$$

and

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{8.12}$$

Let $f_n(z) = x[n]z^{-n}$, $X_+(z)$ is expanded as:

$$\begin{aligned} X_+(z) &= x[0]z^{-0} + x[1]z^{-1} + \cdots + x[n]z^{-n} + \cdots \\ &= f_0(z) + f_1(z) + \cdots + f_n(z) + \cdots \end{aligned} \tag{8.13}$$

According to the ratio test, convergence of $X_+(z)$ requires

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| < 1 \tag{8.14}$$

Let $\lim_{n\to\infty}|x[n+1]/x[n]| = R_+ > 0$. $X_+(z)$ converges if

$$\lim_{n\to\infty}\left|\frac{x[n+1]z^{-n-1}}{x[n]z^{-n}}\right| = \lim_{n\to\infty}\left|\frac{x[n+1]}{x[n]}\right| |z^{-1}| < 1$$

$$\Rightarrow |z| > \lim_{n\to\infty}\left|\frac{x[n+1]}{x[n]}\right| = R_+ \tag{8.15}$$

That is, the ROC for $X_+(z)$ is $|z| > R_+$.

Let $\lim_{m\to\infty} |x[-m]/x[-m-1]| = R_- > 0$. $X_-(z)$ converges if

$$\lim_{m\to\infty} \left|\frac{x[-m-1]z^{m+1}}{x[-m]z^m}\right| = \lim_{m\to\infty} \left|\frac{x[-m-1]}{x[-m]}\right| |z| < 1$$

$$\Rightarrow |z| < \lim_{m\to\infty} \left|\frac{x[-m]}{x[-m-1]}\right| = R_- \qquad (8.16)$$

As a result, the ROC for $X_-(z)$ is $|z| < R_-$.

Combining the results, the ROC for $X(z)$ is $R_+ < |z| < R_-$:

- ROC is a ring when $R_+ < R_-$
- No ROC if $R_- < R_+$ and $X(z)$ does not exist

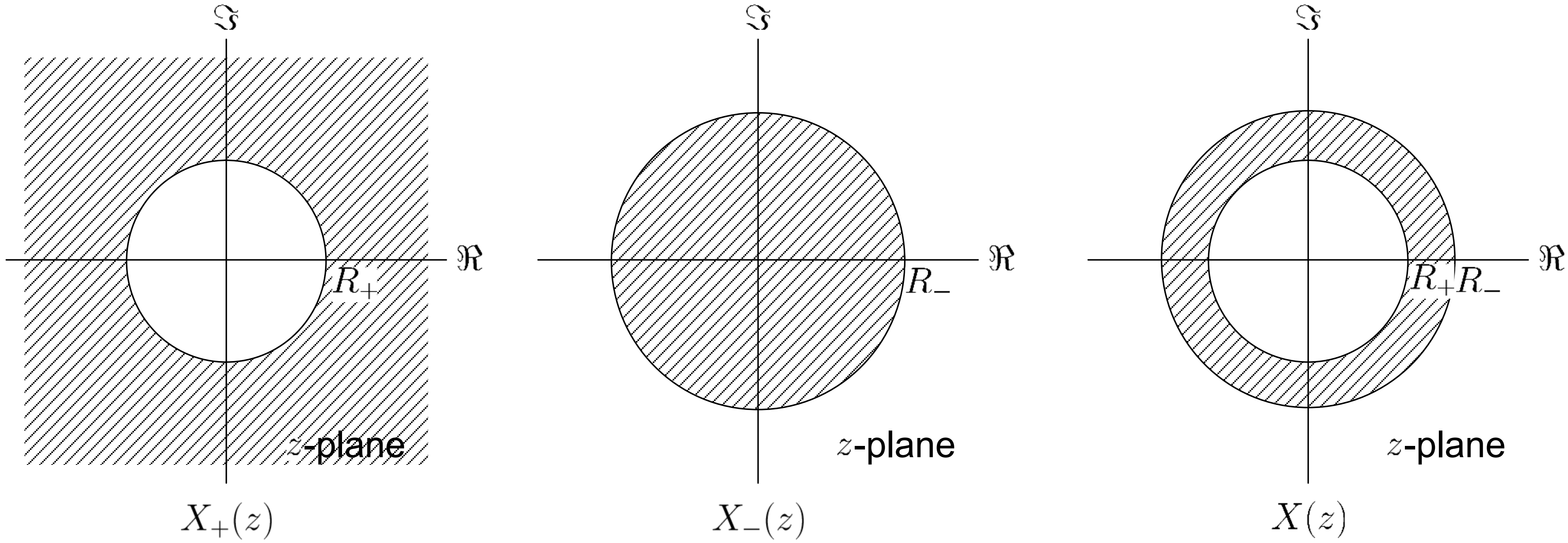

Fig. 8.2: ROCs for $X_+(z)$, $X_-(z)$ and $X(z)$

Poles and Zeros

Values of z for which $X(z) = 0$ are the zeros of $X(z)$.

Values of z for which $X(z) = \pm\infty$ are the poles of $X(z)$.

Example 8.1

In many real-world applications, $X(z)$ is represented as a rational function in z^{-1}:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

Discuss the poles and zeros of $X(z)$.

Multiplying both $P(z)$ and $Q(z)$ by z^{M+N} and then perform factorization yields:

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} = \frac{z^N b_0 (z-d_1)(z-d_2)\cdots(z-d_M)}{z^M a_0 (z-c_1)(z-c_2)\cdots(z-c_N)}$$

We see that there are M nonzero zeros, namely, $d_1, d_2, \cdots, d_M$, and N nonzero poles, namely, $c_1, c_2, \cdots, c_N$.

If $M > N$, there are $(M-N)$ poles at zero location.

On the other hand, if $M < N$, there are $(N-M)$ zeros at zero location.

Note that we use a "$\circ$" to represent a zero and a "$\times$" to represent a pole on the z-plane.

Example 8.2

Determine the z transform of $x[n]=a^n u[n]$ where $u[n]$ is the unit step function. Then determine the condition when the DTFT of $x[n]$ exists.

Using (8.1) and (2.34), we have

$$X(z)=\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

According to (8.7), $X(z)$ converges if

$$\sum_{n=0}^{\infty}\left|az^{-1}\right|^n<\infty$$

Applying the ratio test, the convergence condition is

$$\left|az^{-1}\right|<1 \Leftrightarrow |z|>|a|$$

which aligns with the ROC for $X_+(z)$ in (8.15).

Note that we cannot write $|z| > a$ because a may be complex.

For $|z| > |a|$, $X(z)$ is computed as

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1 - \left(az^{-1}\right)^{\infty}}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Together with the ROC, the z transform of $x[n] = a^n u[n]$ is:

$$X(z) = \frac{z}{z - a}, \quad |z| > |a|$$

It is clear that $X(z)$ has a zero at $z = 0$ and a pole at $z = a$. Using (8.5), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 > |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| < 1$.

Otherwise, its DTFT does not exist. In general, the DTFT $X(e^{j\omega})$ exists if its ROC includes the unit circle. If $|z|>|a|$ includes $|z|=1$, $|a|<1$ is required.

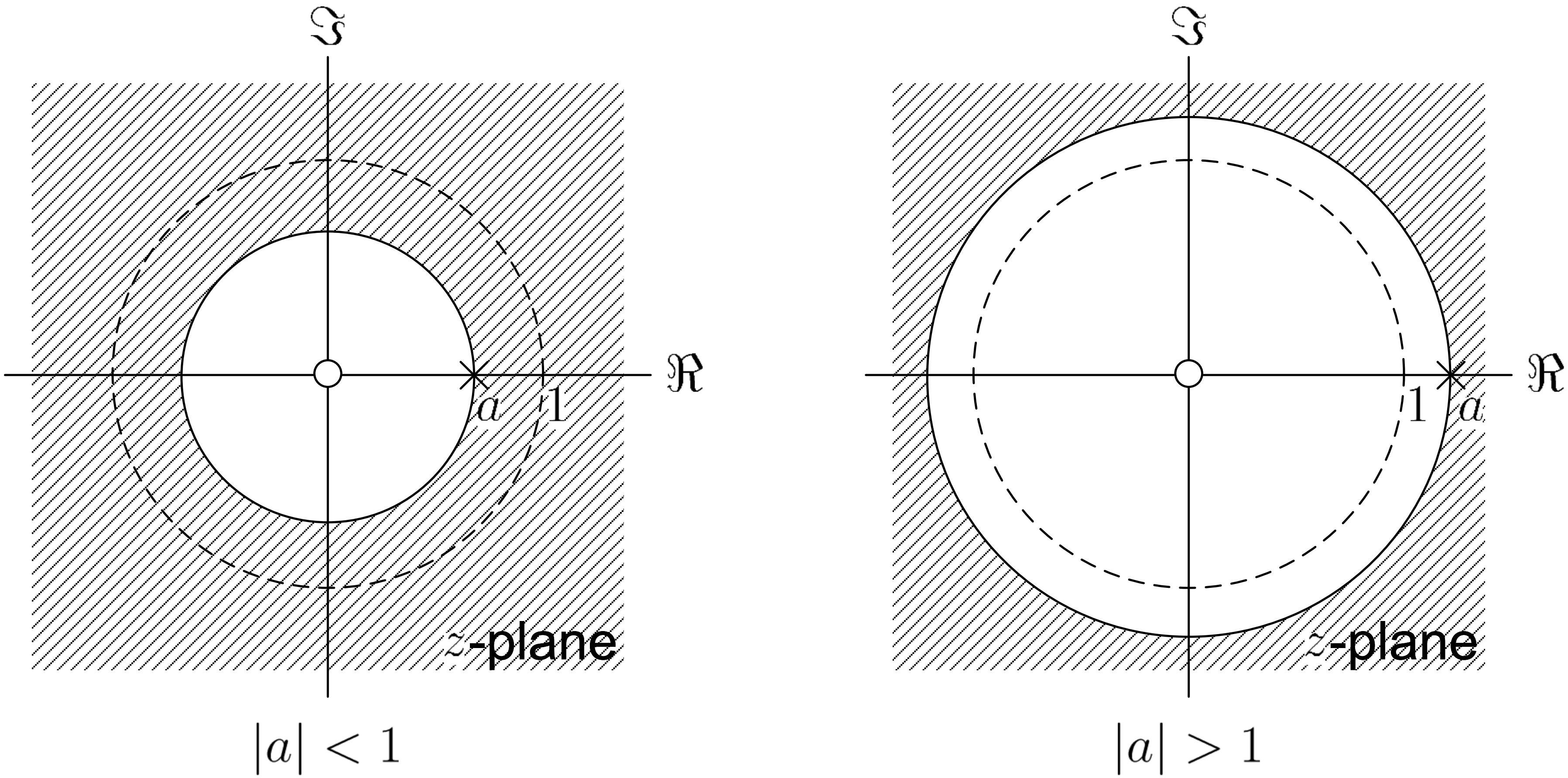

Fig. 8.3: ROCs for $|a|<1$ and $|a|>1$ when $x[n]=a^n u[n]$

Example 8.3

Determine the z transform of $x[n] = -a^n u[-n-1]$. Then determine the condition when the DTFT of $x[n]$ exists.

Using (8.1) and (2.34), we have

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m$$

Similar to Example 8.2, $X(z)$ converges if $\left|a^{-1}z\right| < 1$ or $|z| < |a|$, which aligns with the ROC for $X_-(z)$ in (8.16). This gives

$$X(z) = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m = -\frac{a^{-1}z\left(1 - \left(a^{-1}z\right)^{\infty}\right)}{1 - a^{-1}z} = -\frac{a^{-1}z}{1 - a^{-1}z} = \frac{z}{z-a}$$

Together with ROC, the z transform of $x[n] = -a^n u[-n-1]$ is:

$$X(z) = \frac{z}{z-a}, \quad |z| < |a|$$

Using (8.5), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 < |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| > 1$.

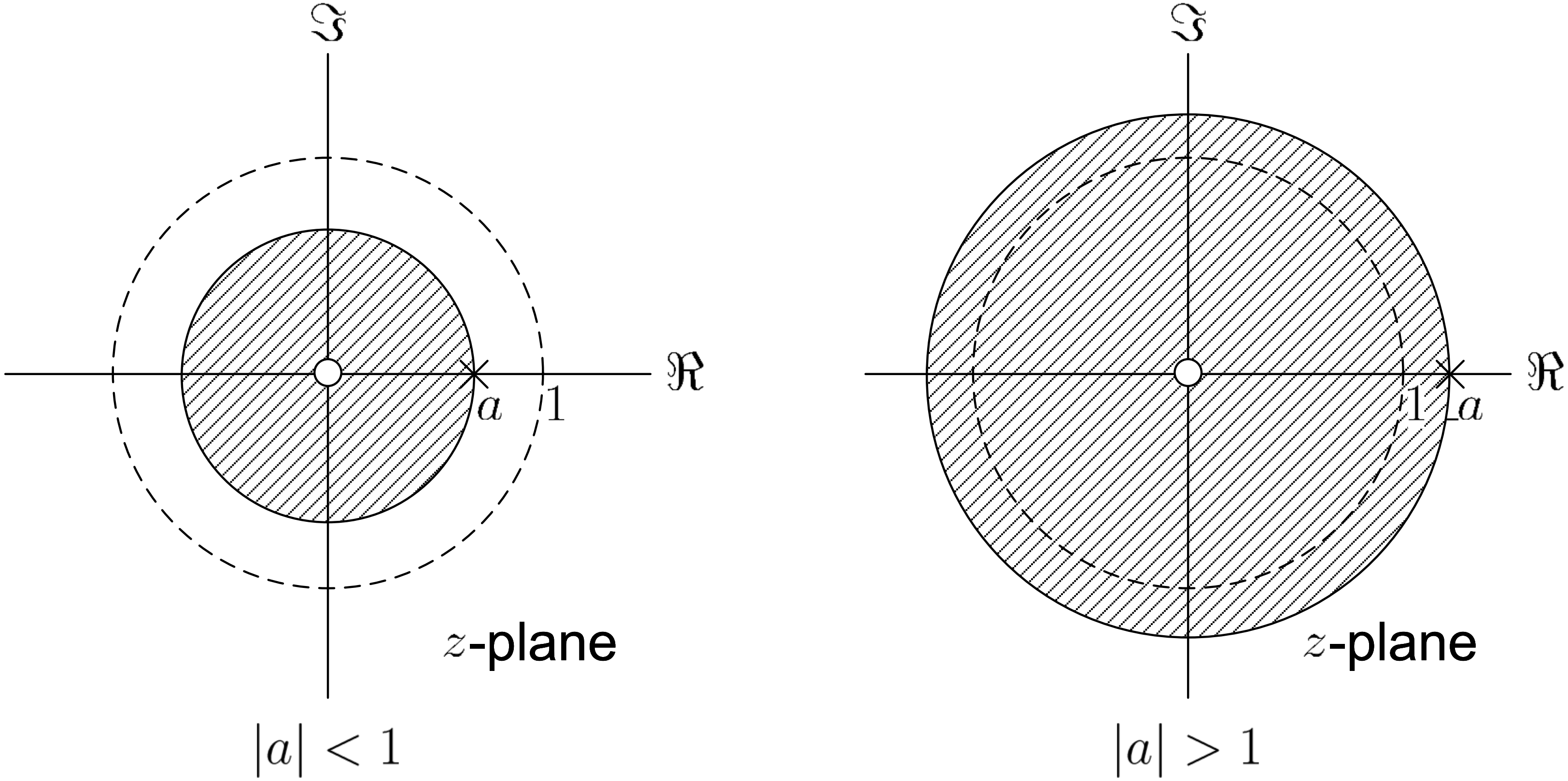

Fig. 8.4: ROCs for $|a| < 1$ and $|a| > 1$ when $x[n] = -a^n u[-n-1]$

Example 8.4

Determine the z transform of $x[n] = a^n u[n] + b^n u[-n-1]$ where $|a| < |b|$.

Employing the results in Examples 8.2 and 8.3, we have

$$
\begin{aligned}
X(z) &= \frac{1}{1-az^{-1}} + \left(-\frac{1}{1-bz^{-1}}\right), \quad |z| > |a| \quad \text{and} \quad |z| < |b| \\
&= \frac{(a-b)z^{-1}}{(1-az^{-1})(1-bz^{-1})} \\
&= \frac{(a-b)z}{(z-a)(z-b)}, \quad |a| < |z| < |b|
\end{aligned}
$$

Note that its ROC agrees with Fig. 8.2.

What are the pole(s) and zero(s) of *X*(*z*)?

Example 8.5

Determine the z transform of $x[n] = \delta[n+1]$.

Using (8.1) and (2.33), we have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z$$

Example 8.6

Determine the z transform of $x[n]$ which has the form of:

$$x[n] = \begin{cases} a^n, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases}$$

Using (8.1), we have

$$X(z) = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

What are the ROCs in Examples 8.5 and 8.6?

Finite-Duration and Infinite-Duration Sequences

Finite-duration sequence: values of $x[n]$ are nonzero only for a finite time interval.

Otherwise, $x[n]$ is called an infinite-duration sequence:

- Right-sided: if $x[n]=0$ for $n < N_+ < \infty$ where N_+ is an integer (e.g., $x[n]=a^n u[n]$ with $N_+=0$; $x[n]=a^n u[n-10]$ with $N_+=10$; $x[n]=a^n u[n+10]$ with $N_+=-10$).
- Left-sided: if $x[n]=0$ for $n > N_- > -\infty$ where N_- is an integer (e.g., $x[n]=-a^n u[-n-1]$ with $N_-=-1$).
- Two-sided: neither right-sided nor left-sided (e.g., Example 8.4).

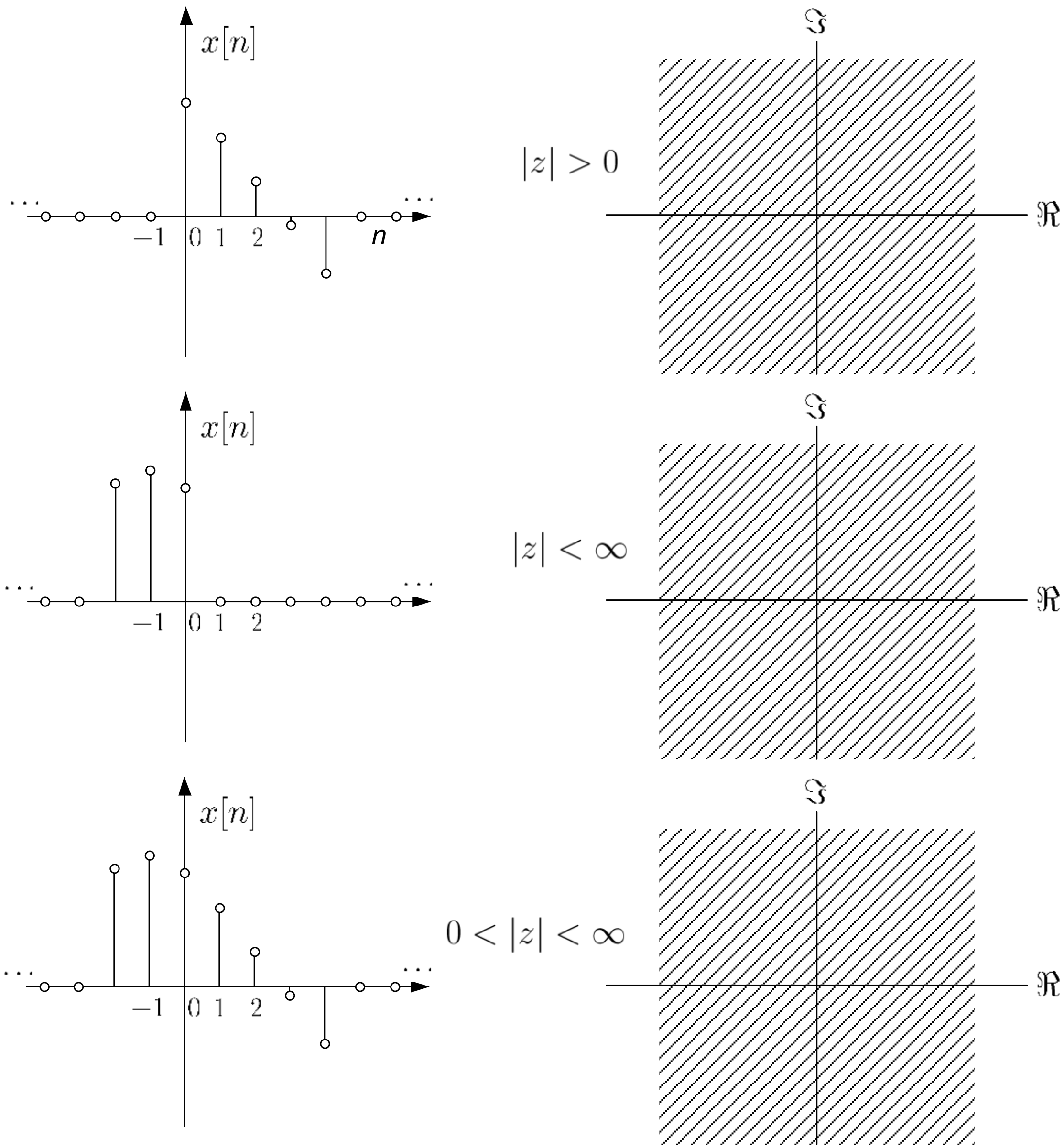

Fig. 8.5: Finite-duration sequences

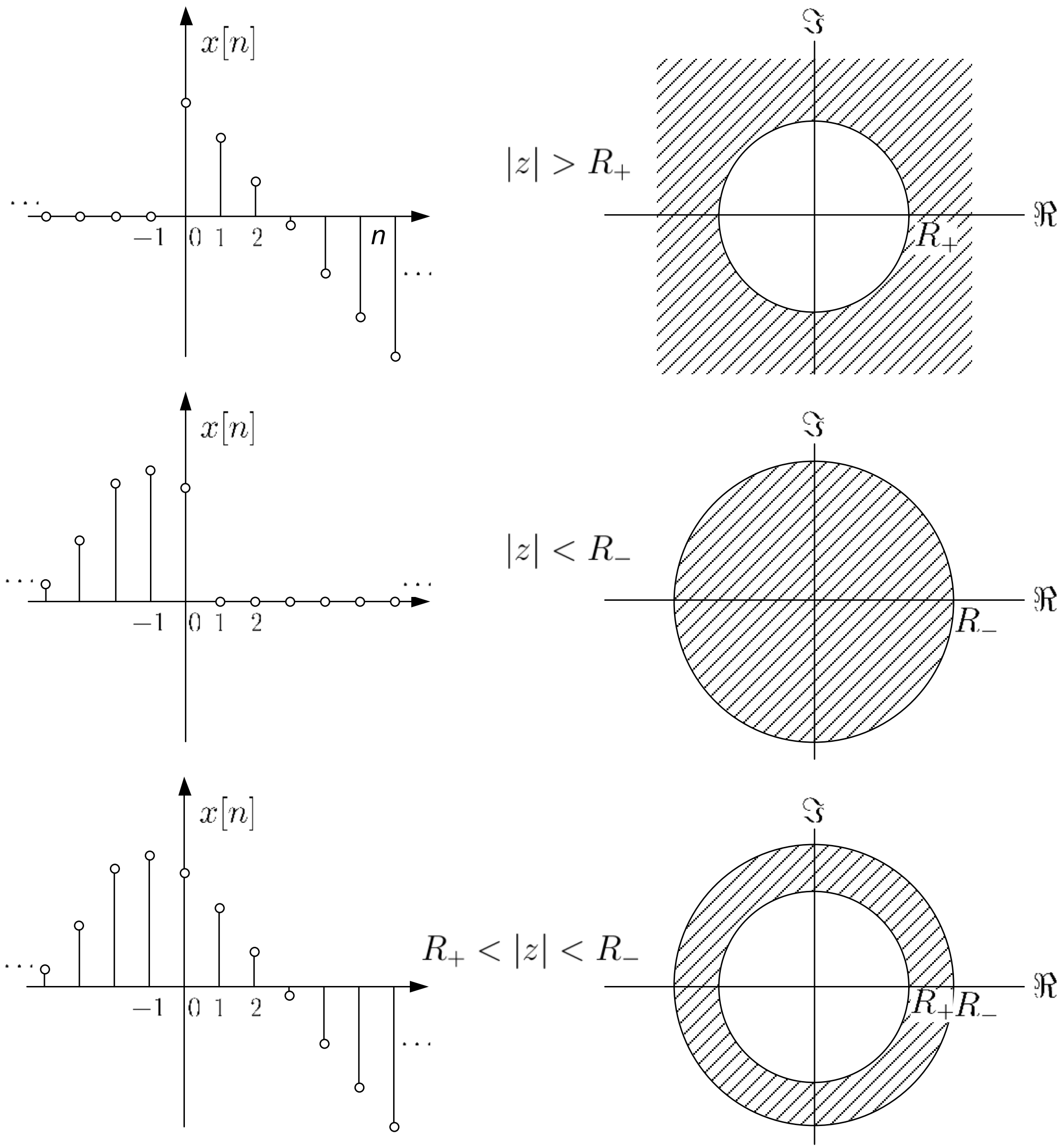

Fig. 8.6: Infinite-duration sequences

Sequence	Transform	ROC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	$\|z\|>0$, $m>0$; $\|z\|<\infty$, $m<0$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\|>\|a\|$
$-a^n u[-n-1]$	$\frac{1}{1-az^{-1}}$	$\|z\|<\|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|>\|a\|$
$-na^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\|<\|a\|$
$a^n \cos(bn) u[n]$	$\frac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$
$a^n \sin(bn) u[n]$	$\frac{a\sin(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\|z\|>\|a\|$

Table 8.1: z transforms for common sequences

Summary of ROC Properties

P1. There are four possible shapes for ROC, namely, the entire region except possibly $z = 0$ and/or $z = \infty$, a ring, or inside or outside a circle in the z-plane centered at the origin (e.g., Figures 8.6 and 8.7).

P2. The DTFT of a sequence $x[n]$ exists if and only if the ROC of the z transform of $x[n]$ includes the unit circle (e.g., Examples 8.2 and 8.3).

P3: The ROC cannot contain any poles (e.g., Examples 8.2 to 8.4).

P4: When $x[n]$ is a finite-duration sequence, the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \infty$ (e.g., Examples 8.5 and 8.6).

P5: When $x[n]$ is a right-sided sequence, the ROC is of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the pole with the largest magnitude in $X(z)$ (e.g., Example 8.2).

P6: When $x[n]$ is a left-sided sequence, the ROC is of the form $|z| < |p_{\min}|$ where $p_{\min}$ is the pole with the smallest magnitude in $X(z)$ (e.g., Example 8.3).

P7: When $x[n]$ is a two-sided sequence, the ROC is of the form $|p_a| < |z| < |p_b|$ where p_a and p_b are two poles with the successive magnitudes in $X(z)$ such that $|p_a| < |p_b|$ (e.g., Example 8.4).

P8: The ROC must be a connected region.

<u>Example 8.7</u>

A z transform $X(z)$ contains three poles, namely, a, b and c with $|a| < |b| < |c|$. Determine all possible ROCs.

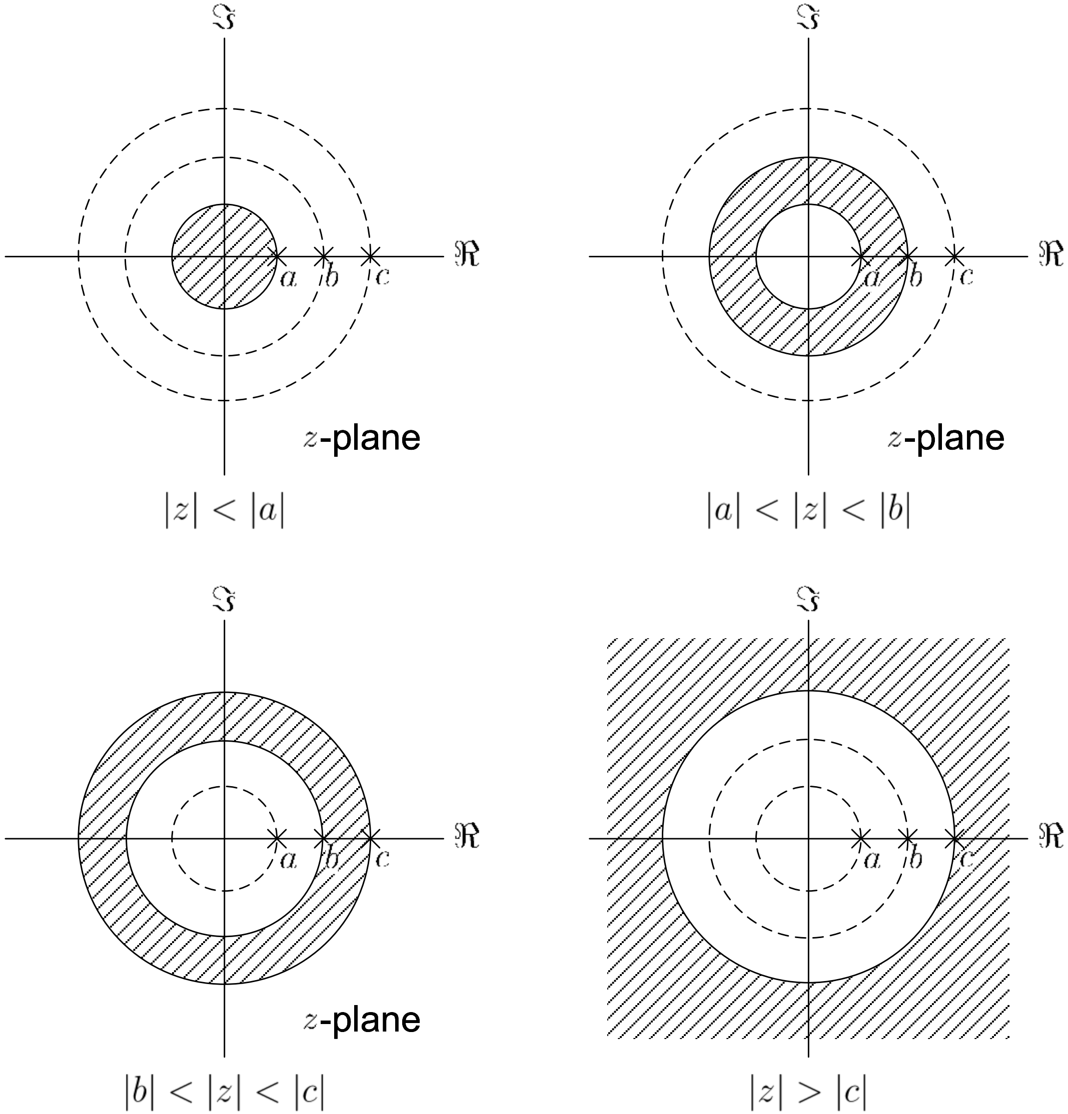

Fig. 8.7: ROC possibilities for three poles

What are other possible ROCs?

Properties of z Transform

Linearity

Let $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, we have

$$a x_1[n] + b x_2[n] \leftrightarrow a X_1(z) + b X_2(z) \tag{8.17}$$

Its ROC is denoted by $\mathcal{R}$, which includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$ where $\cap$ is the intersection operator. That is, $\mathcal{R}$ contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Example 8.8

Determine the z transform of $y[n]$ which is expressed as:

$$y[n] = x_1[n] + x_2[n]$$

where $x_1[n] = (0.2)^n u[n]$ and $x_2[n] = (-0.3)^n u[n]$.

From Table 8.1, the z transforms of $x_1[n]$ and $x_2[n]$ are:

$$x_1[n] = (0.2)^n u[n] \leftrightarrow \frac{1}{1-0.2z^{-1}}, \quad |z| > 0.2$$

and

$$x_2[n] = (-0.3)^n u[n] \leftrightarrow \frac{1}{1+0.3z^{-1}}, \quad |z| > 0.3$$

According to the linearity property, the z transform of $y[n]$ is

$$Y(z) = \frac{1}{1-0.2z^{-1}} + \frac{1}{1+0.3z^{-1}}, \quad |z| > 0.3$$

Why the ROC is |z|>0.3 instead of |z|>0.2?

Example 8.9

Determine the ROC of the z transform of $x[n]$ which is expressed as:

$$x[n] = a^n u[n] - a^n u[n-1]$$

Noting that $a^n u[n] - a^n u[n-1] = \delta[n]$, we know that the ROC of $x[n]$ is the entire z-plane.

On the other hand, both ROCs of $a^n u[n]$ and $a^n u[n-1]$ are $|z| > |a|$. We see that the ROC of $x[n]$ contains the intersections of $a^n u[n]$ and $a^n u[n-1]$, which is $|z| > |a|$.

Time Shifting

A time-shift of n_0 in $x[n]$ causes a multiplication of z^{-n_0} in $X(z)$

$$x[n-n_0] \leftrightarrow z^{-n_0} X(z) \qquad (8.18)$$

The ROC for $x[n-n_0]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z=0$ or $z=\infty$.

Example 8.10

Find the z transform of $x[n]$ which has the form of:

$$x[n] = a^{n-1}u[n-1]$$

Employing the time shifting property with $n_0 = 1$ and:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

we easily obtain

$$a^{n-1}u[n-1] \leftrightarrow z^{-1} \cdot \frac{1}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}, \quad |z| > |a|$$

Note that using (8.1) with $|z| > |a|$ also produces the same result but this approach is less efficient:

$$X(z) = \sum_{n=1}^{\infty} a^{n-1} z^{-n} = a^{-1} \sum_{n=1}^{\infty} \left(az^{-1}\right)^n = a^{-1} \frac{az^{-1}\left[1-\left(az^{-1}\right)^{\infty}\right]}{1-az^{-1}} = \frac{z^{-1}}{1-az^{-1}}$$

Multiplication by an Exponential Sequence

If we multiply $x[n]$ by z_0^n in the time domain, the variable z will be changed to z/z_0 in the z transform domain. That is:

$$z_0^n x[n] \leftrightarrow X(z/z_0) \tag{8.19}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, then the ROC for $z_0^n x[n]$ is $|z_0|R_+ < |z| < |z_0|R_-$.

Example 8.11

With the use of the following z transform pair:

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

Find the z transform of $x[n]$ which has the form of:

$$x[n] = a^n \cos(bn) u[n]$$

Noting that $\cos(bn) = (e^{jbn} + e^{-jbn})/2$, $x[n]$ can be written as:

$$x[n] = \frac{1}{2}\left(ae^{jb}\right)^n u[n] + \frac{1}{2}\left(ae^{-jb}\right)^n u[n]$$

By means of the property of (8.19) with the substitution of $z_0 = ae^{jb}$ and $z_0 = ae^{-jb}$, we obtain:

$$\frac{1}{2}\left(ae^{jb}\right)^n u[n] \leftrightarrow \frac{1}{2}\frac{1}{1-(z/(ae^{jb}))^{-1}} = \frac{1}{2}\frac{1}{1-ae^{jb}z^{-1}}, \quad |z| > |a|$$

and

$$\frac{1}{2}\left(ae^{-jb}\right)^n u[n] \leftrightarrow \frac{1}{2}\frac{1}{1-(z/(ae^{-jb}))^{-1}} = \frac{1}{2}\frac{1}{1-ae^{-jb}z^{-1}}, \quad |z| > |a|$$

By means of the linearity property, it follows that

$$X(z) = \frac{1}{2}\frac{1}{1-ae^{jb}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-jb}z^{-1}} = \frac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}, |z| > |a|$$

which agrees with Table 8.1.

Differentiation

Differentiating $X(z)$ with respect to z corresponds to multiplying $x[n]$ by n in the time domain:

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \tag{8.20}$$

The ROC for $nx[n]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z=0$ or $z=\infty$.

Example 8.12

Determine the z transform of $x[n]=na^n u[n]$.

We have:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z|>|a|$$

and

$$\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) = \frac{d\left(1-az^{-1}\right)^{-1}}{d\left(1-az^{-1}\right)} \cdot \frac{d\left(1-az^{-1}\right)}{dz} = -\frac{az^{-2}}{\left(1-az^{-1}\right)^2}$$

By means of the differentiation property, we obtain:

$$na^n u[n] \leftrightarrow -z \cdot -\frac{az^{-2}}{\left(1-az^{-1}\right)^2} = \frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z| > |a|$$

which agrees with Table 8.1.

Conjugation

The z transform pair for $x^*[n]$ is:

$$x^*[n] \leftrightarrow X^*(z^*) \tag{8.21}$$

The ROC for $x^*[n]$ is identical to that of $x[n]$.

Time Reversal

The z transform pair for $x[-n]$ is:

$$x[-n] \leftrightarrow X(z^{-1}) \tag{8.22}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, the ROC for $x[-n]$ is $1/R_- < |z| < 1/R_+$.

Example 8.13

Determine the z transform of $x[n] = -na^{-n}u[-n]$.

Using Example 8.12:

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

and from the time reversal property:

$$X(z) = \frac{az}{(1-az)^2} = \frac{a^{-1}z^{-1}}{(1-a^{-1}z^{-1})^2}, \quad |z| < |a^{-1}|$$

Convolution

Let $x_1[n] \leftrightarrow X_1(z)$ and $x_2[n] \leftrightarrow X_2(z)$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively. Then we have:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad (8.23)$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$.

The proof is given as follows.

Let

$$y[n] = x_1[n] \otimes x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \qquad (8.24)$$

With the use of the time shifting property, $Y(z)$ is:

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \right] \\
&= \sum_{k=-\infty}^{\infty} x_1[k] X_2(z) z^{-k} \\
&= X_1(z) X_2(z) \qquad (8.25)
\end{aligned}
$$

Causality and Stability Investigation with ROC

Suppose $h[n]$ is the impulse response of a discrete-time linear time-invariant (LTI) system. Recall (3.19), which is the causality condition:

$$h[n] = 0, \quad n < 0 \tag{8.26}$$

If the system is causal and $h[n]$ is of finite duration, the ROC should include ∞ (See Example 8.5 and Figure 8.5).

If the system is causal and $h[n]$ is of infinite duration, the ROC is of the form $|z| > |p_{\max}|$ and should include ∞ (See Example 8.2 and Figure 8.6). According to P5, $h[n]$ must be a right-sided sequence.

Example 8.14

Consider a LTI system with impulse response $h[n]$:

$$h[n] = a^{n+10} u[n+10]$$

Discuss the causality of the system.

According to (8.26), the system is not causal. Although it is a right-sided sequence, the ROC of $H(z)$ does not include ∞:

$$H(z) = \sum_{n=-\infty}^{\infty} a^{n+10} u[n+10] z^{-n} = a^{10} \left(\left(\frac{a}{z}\right)^{-10} + \left(\frac{a}{z}\right)^{-9} + \cdots \right)$$

where z cannot be equal to ∞ for convergence.

Applying the time shifting property, we get:

$$a^{n+10}u[n+10] \leftrightarrow z^{10} \cdot \frac{1}{1-az^{-1}} = \frac{z^{10}}{1-az^{-1}} = \frac{z^{11}}{z-a}, \quad |z|>|a|$$

The numerator has degree 11 while the denominator has degree 1, making the ROC cannot include ∞.

Generalizing the results, for a rational $H(z)$, it will be a causal system if its ROC has the form of $|z|>|p_{\max}|$ and the order of the numerator is not greater than that of the denominator.

Recall the stability condition in (3.21):

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty \qquad (8.27)$$

Based on (8.9), this also means that the DTFT of $h[n]$ exists.

According to P2, (8.27) indicates that the ROC of $H(z)$ should include the unit circle.

<u>Example 8.15</u>

Consider a LTI system with impulse response $h[n]$:

$$h[n] = a^{n+10} u[n+10]$$

Discuss the stability of the system.

Using the result in Example 8.14, we have:

$$H(z) = \frac{z^{10}}{1 - az^{-1}}, \quad |z| > |a|$$

That is, if $|a| < 1$, then the system is stable. Otherwise, the system is not stable.

Inverse z Transform

Inverse z transform corresponds to finding $x[n]$ given $X(z)$ and its ROC.

The z transform and inverse z transform are one-to-one mapping provided that the ROC is given:

$$x[n] \leftrightarrow X(z) \tag{8.28}$$

There are 4 commonly used techniques to evaluate the inverse z transform. They are

1. Inspection
2. Partial Fraction Expansion
3. Power Series Expansion
4. Cauchy Integral Theorem

Inspection

When we are familiar with certain transform pairs, we can do the inverse z transform by inspection.

Example 8.16

Determine the inverse z transform of $X(z)$ which is expressed as:

$$X(z) = \frac{z}{2z-1}, \quad |z| > 0.5$$

We first rewrite $X(z)$ as:

$$X(z) = \frac{0.5}{1-0.5z^{-1}}$$

Making use of the following transform pair in Table 8.1:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

and putting $a = 0.5$, we have:

$$\frac{0.5}{1-0.5z^{-1}} \leftrightarrow 0.5(0.5)^n u[n]$$

By inspection, the inverse z transform is:

$$x[n] = (0.5)^{n+1} u[n]$$

Partial Fraction Expansion

We consider that $X(z)$ is a rational function in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \quad (8.29)$$

To obtain the partial fraction expansion from (8.29), the first step is to determine the N nonzero poles, $c_1, c_2, \cdots, c_N$.

There are 4 cases to be considered:

Case 1: $M < N$ and all poles are of first order

For first-order poles, all $\{c_k\}$ are distinct. $X(z)$ is:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \qquad (8.30)$$

For each first-order term of $A_k / \left(1 - c_k z^{-1}\right)$, its inverse z transform can be easily obtained by inspection.

Multiplying both sides by $\left(1 - c_k z^{-1}\right)$ and evaluating for $z = c_k$

$$A_k = \left(1 - c_k z^{-1}\right) X(z)\Big|_{z=c_k} \qquad (8.31)$$

An illustration for computing A_1 with $N = 2 > M$ is:

$$X(z) = \frac{A_1}{1 - c_1 z^{-1}} + \frac{A_2}{1 - c_2 z^{-1}}$$
$$\Rightarrow \left(1 - c_1 z^{-1}\right) X(z) = A_1 + \frac{A_2 \left(1 - c_1 z^{-1}\right)}{1 - c_2 z^{-1}} \qquad (8.32)$$

Substituting $z = c_1$, we get A_1.

In summary, three steps are:

- Find poles.
- Find $\{A_k\}$.
- Perform inverse z transform for the fractions by inspection.

Example 8.17

Find the pole and zero locations of $H(z)$:

$$H(z) = -\frac{1 + 0.1z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

Then determine the inverse z transform of $H(z)$.

We first multiply z^2 to both numerator and denominator polynomials to obtain:

$$H(z) = -\frac{z\left(z + 0.1\right)}{z^2 - 2.05z + 1}$$

Apparently, there are two zeros at $z = 0$ and $z = -0.1$. On the other hand, by solving the quadratic equation at the denominator polynomial, the poles are determined as $z = 0.8$ and $z = 1.25$.

According to (8.30), we have:

$$H(z)=\frac{A_1}{1-0.8z^{-1}}+\frac{A_2}{1-1.25z^{-1}}$$

Employing (8.31), A_1 is calculated as:

$$A_1=\left(1-0.8z^{-1}\right)H(z)\big|_{z=0.8}=-\left.\frac{1+0.1z^{-1}}{1-1.25z^{-1}}\right|_{z=0.8}=2$$

Similarly, A_2 is found to be -3. As a result, the partial fraction expansion for $H(z)$ is

$$H(z)=\frac{2}{1-0.8z^{-1}}-\frac{3}{1-1.25z^{-1}}$$

As the ROC is not specified, we investigate all possible scenarios, namely, $|z|>1.25$, $0.8<|z|<1.25$, and $|z|<0.8$.

For $|z| > 1.25$, we notice that

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1 - 0.8z^{-1}}, \quad |z| > 0.8$$

and

$$(1.25)^n u[n] \leftrightarrow \frac{1}{1 - 1.25z^{-1}}, \quad |z| > 1.25$$

where both ROCs agree with $|z| > 1.25$. Combining the results, the inverse z transform $h[n]$ is:

$$h[n] = (2(0.8)^n - 3(1.25)^n)\, u[n]$$

which is a right-sided sequence and aligns with P5.

For $0.8 < |z| < 1.25$, we make use of

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1 - 0.8z^{-1}}, \quad |z| > 0.8$$

and

$$-(1.25)^n u[-n-1] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| < 1.25$$

where both ROCs agree with $0.8 < |z| < 1.25$. This implies:

$$h[n] = 2(0.8)^n u[n] + 3(1.25)^n u[-n-1]$$

which is a two-sided sequence and aligns with P7.

Finally, for $|z| < 0.8$:

$$-(0.8)^n u[-n-1] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| < 0.8$$

and

$$-(1.25)^n u[-n-1] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| < 1.25$$

where both ROCs agree with $|z| < 0.8$. As a result, we have:

$$h[n] = \left(-2(0.8)^n + 3(1.25)^n\right) u[-n-1]$$

which is a left-sided sequence and aligns with P6.

Suppose $h[n]$ is the impulse response of a discrete-time LTI system.

In terms of causality and stability, there are three possible cases:

- $h[n] = (2(0.8)^n - (1.25)^n)\,u[n]$ is the impulse response of a causal but unstable system (ROC: $|z| > 1.25$).

- $h[n] = 2(0.8)^n u[n] + (1.25)^n u[-n-1]$ corresponds to a non-causal but stable system (ROC: $0.8 < |z| < 1.25$).

- $h[n] = (-2(0.8)^n + (1.25)^n)\,u[-n-1]$ is non-causal and unstable (ROC: $|z| < 0.8$).

Case 2: $M \geq N$ and all poles are of first order

In this case, $X(z)$ can be expressed as:

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \qquad (8.33)$$

- B_l are obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.
- A_k can be obtained using (8.31).

Example 8.18

Determine $x[n]$ which has z transform of the form:

$$X(z) = \frac{4 - 2z^{-1} + z^{-2}}{1 - 1.5z^{-1} + 0.5z^{-2}}, \quad |z| > 1$$

The poles are easily determined as $z = 0.5$ and $z = 1$

According to (8.33) with $M = N = 2$:

$$X(z) = B_0 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

The value of B_0 is found by dividing the numerator polynomial by the denominator polynomial as follows:

$$\begin{array}{r|l} & \quad 2 \\ \hline 0.5z^{-2} - 1.5z^{-1} + 1 & z^{-2} - 2z^{-1} + 4 \\ & \underline{z^{-2} - 3z^{-1} + 2} \\ & \qquad\quad z^{-1} + 2 \end{array}$$

That is, $B_0 = 2$. Thus $X(z)$ is expressed as

$$X(z) = 2 + \frac{2 + z^{-1}}{(1 - 0.5z^{-1})(1 - z^{-1})} = 2 + \frac{A_1}{1 - 0.5z^{-1}} + \frac{A_2}{1 - z^{-1}}$$

According to (8.31), A_1 and A_2 are calculated as

$$A_1 = \left. \frac{4 - 2z^{-1} + z^{-2}}{1 - z^{-1}} \right|_{z=0.5} = -4$$

and

$$A_2 = \left. \frac{4 - 2z^{-1} + z^{-2}}{1 - 0.5z^{-1}} \right|_{z=1} = 6$$

With $|z| > 1$:

$$\delta[n] \leftrightarrow 1$$

$$(0.5)^n u[n] \leftrightarrow \frac{1}{1 - 0.5z^{-1}}, \quad |z| > 0.5$$

and

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

the inverse z transform $x[n]$ is:

$$x[n] = 2\delta[n] - 4(0.5)^n u[n] + 6u[n]$$

Case 3: $M < N$ with multiple-order pole(s)

If $X(z)$ has a s-order pole at $z = c_i$ with $s \geq 2$, this means that there are s repeated poles with the same value of c_i. $X(z)$ is:

$$X(z) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - c_i z^{-1}\right)^m} \qquad (8.34)$$

- When there are two or more multiple-order poles, we include a component like the second term for each corresponding pole
- A_k can be computed according to (8.31)
- C_m can be calculated from:

$$C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} \left[(1 - c_i w)^s X(w^{-1}) \right] \Big|_{w=c_i^{-1}} \qquad (8.35)$$

Example 8.19

Determine the partial fraction expansion for $X(z)$:

$$X(z) = \frac{4}{\left(1+z^{-1}\right)\left(1-z^{-1}\right)^2}$$

It is clear that $X(z)$ corresponds to Case 3 with $N = 3 > M$ and one second-order pole at $z = 1$. Hence $X(z)$ is:

$$X(z) = \frac{A_1}{1+z^{-1}} + \frac{C_1}{1-z^{-1}} + \frac{C_2}{\left(1-z^{-1}\right)^2}$$

Employing (8.31), A_1 is:

$$A_1 = \left.\frac{4}{\left(1-z^{-1}\right)^2}\right|_{z=-1} = 1$$

Applying (8.35), C_1 is:

$$
\begin{aligned}
C_1 &= \frac{1}{(2-1)!(-1)^{2-1}} \cdot \frac{d}{dw}\left[(1-1\cdot w)^2 \frac{4}{(1+w)(1-w)^2}\right]\Bigg|_{w=1} \\
&= -\frac{d}{dw}\frac{4}{1+w}\bigg|_{w=1} \\
&= \frac{4}{(1+w)^2}\bigg|_{w=1} \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 &= \frac{1}{(2-2)!(-1)^{2-2}} \cdot \left[(1-1\cdot w)^2 \frac{4}{(1+w)(1-w)^2}\right]\Bigg|_{w=1} \\
&= \frac{4}{1+w}\bigg|_{w=1} \\
&= 2
\end{aligned}
$$

Therefore, the partial fraction expansion for $X(z)$ is

$$X(z)=\frac{1}{1+z^{-1}}+\frac{1}{1-z^{-1}}+\frac{2}{\left(1-z^{-1}\right)^{2}}$$

Case 4: $M \geq N$ with multiple-order pole(s)

This is the most general case and the partial fraction expansion of $X(z)$ is

$$X(z)=\sum_{l=0}^{M-N} B_l z^{-l}+\sum_{k=1,k\neq i}^{N}\frac{A_k}{1-c_k z^{-1}}+\sum_{m=1}^{s}\frac{C_m}{\left(1-c_i z^{-1}\right)^{m}} \quad (8.36)$$

assuming that there is only one multiple-order pole of order $s \geq 2$ at $z=c_i$. It is easily extended to the scenarios when there are two or more multiple-order poles as in Case 3. The A_k , B_l and C_m can be calculated as in Cases 1, 2 and 3.

Power Series Expansion

When $X(z)$ is expanded as power series according to (8.1):

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^1 + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \quad (8.37)$$

any particular value of $x[n]$ can be determined by finding the coefficient of the appropriate power of z^{-1}.

Example 8.20

Determine $x[n]$ which has z transform of the form:

$$X(z) = 2z^2 \left(1 - 0.5z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right), \quad 0 < |z| < \infty$$

Expanding $X(z)$ yields

$$X(z) = 2z^2 - z - 2 + z^{-1}$$

From (8.37), $x[n]$ is deduced as:

$$x[n] = 2\delta[n+2] - \delta[n+1] - 2\delta[n] + \delta[n-1]$$

Example 8.21

Determine $x[n]$ whose z transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

With the use of

$$\frac{1}{1-\lambda} = 1 + \lambda + \lambda^2 + \cdots, \quad |\lambda| < 1$$

Carrying out long division in $X(z)$ with $|az^{-1}| < 1$:

$$X(z) = 1 + az^{-1} + \left(az^{-1}\right)^2 + \cdots$$

From (8.37), $x[n]$ is deduced as:

$$x[n] = a^n u[n]$$

which agrees with Example 8.2 and Table 8.1.

Example 8.22

Determine $x[n]$ whose z transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

We first express $X(z)$ as:

$$X(z) = \frac{-a^{-1}z}{-a^{-1}z} \cdot \frac{1}{1 - az^{-1}} = \frac{-a^{-1}z}{1 - a^{-1}z}$$

Carrying out long division in $X(z)$ with $|a^{-1}z| < 1$:

$$X(z) = -a^{-1}z\left(1 + a^{-1}z + \left(a^{-1}z\right)^2 + \cdots\right) = -\left(a^{-1}z + a^{-2}z^2 + \cdots\right)$$

From (8.37), $x[n]$ is deduced as:

$$x[n] = -a^n u[-n-1]$$

which agrees with Example 8.3 and Table 8.1.

Transfer Function of Linear Time-Invariant System

A LTI system can be characterized by the transfer function, which is a z transform expression.

Starting with:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \quad (8.38)$$

Applying z transform on (8.38) with the use of the linearity and time shifting properties, we have:

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \quad (8.39)$$

The transfer function, denoted by $H(z)$, is defined as:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{8.40}$$

The system impulse response $h[n]$ is given by the inverse z transform of $H(z)$ with an appropriate ROC, that is, $h[n] \leftrightarrow H(z)$, such that $y[n]=x[n]\otimes h[n]$. This suggests that we can first take the z transforms for $x[n]$ and $h[n]$, then multiply $X(z)$ by $H(z)$, and finally perform the inverse z transform of $X(z)H(z)$.

Comparing with (6.26), we see that the system frequency response can be obtained as $H(z)|_{z=e^{j\omega}}=H(e^{j\omega})$ if it exists.

Example 8.23

Determine the transfer function for a LTI system whose input $x[n]$ and output $y[n]$ are related by:

$$y[n] = 0.1y[n-1] + x[n] + x[n-1]$$

Applying z transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z)\left(1 - 0.1z^{-1}\right) = X(z)\left(1 + z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 0.1z^{-1}}$$

Note that there are two ROC possibilities, namely, $|z| > 0.1$ and $|z| < 0.1$, and we cannot uniquely determine $h[n]$. However, if it is known that the system is causal, $h[n]$ can be uniquely found because the ROC should be $|z| > 0.1$.

Example 8.24

Find the difference equation of a LTI system whose transfer function is given by

$$H(z) = \frac{\left(1 + z^{-1}\right)\left(1 - 2z^{-1}\right)}{\left(1 - 0.5z^{-1}\right)\left(1 + 2z^{-1}\right)}$$

Let $H(z) = Y(z)/X(z)$. Performing cross-multiplication and inverse z transform, we obtain:

$$\left(1 - 0.5z^{-1}\right)\left(1 + 2z^{-1}\right)Y(z) = \left(1 + z^{-1}\right)\left(1 - 2z^{-1}\right)X(z)$$
$$\Rightarrow \left(1 + 1.5z^{-1} - z^{-2}\right)Y(z) = \left(1 - z^{-1} - 2z^{-2}\right)X(z)$$
$$\Rightarrow y[n] + 1.5y[n-1] - y[n-2] = x[n] - x[n-1] - 2x[n-2]$$

Examples 8.23 and 8.24 imply the equivalence between the difference equation and transfer function.

Example 8.25

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n-1]$$

Applying z transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z) = X(z)\left(1 - z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}$$

There is only one ROC possibility, namely, $|z| > 0$. Taking the inverse z transform on $H(z)$, we get:

$$h[n] = \delta[n] - \delta[n-1]$$

which agrees with Example 3.18.

Example 8.26

Determine the output $y[n]$ if the input is $x[n]=u[n]$ and the LTI system impulse response is $h[n]=\delta[n]+0.5\delta[n-1]$

The z transforms for $x[n]$ and $h[n]$ are

$$X(z)=\frac{1}{1-z^{-1}}, \quad |z|>1$$

and

$$H(z)=1+0.5z^{-1} \quad |z|>0$$

As a result, we have:

$$Y(z)=X(z)H(z)=\frac{1}{1-z^{-1}}+0.5\frac{z^{-1}}{1-z^{-1}}, \quad |z|>1$$

Taking the inverse z transform of $Y(z)$ with the use of the time shifting property yields:

$$y[n]=u[n]+0.5u[n-1]$$

which agrees with Example 3.13.