Laplace Transform

Chapter Intended Learning Outcomes:

(i) Represent continuous-time signals using Laplace transform

(ii) Understand the relationship between Laplace transform and Fourier transform

(iii) Understand the properties of Laplace transform

(iv) Perform operations on Laplace transform and inverse Laplace transform

(v) Apply Laplace transform for analyzing linear time-invariant systems

Analog Signal Representation with Laplace Transform

Apart from Fourier transform, we can also use Laplace transform to represent continuous-time signals.

The Laplace transform of $x(t)$, denoted by $X(s)$, is defined as:

$$X(s) = \int_{-\infty}^{\infty} x(t)e^{-st}dt \tag{9.1}$$

where s is a continuous complex variable.

We can also express s as:

$$s = \sigma + j\Omega \tag{9.2}$$

where σ and Ω are the real and imaginary parts of s, respectively.

Employing (9.2), the Laplace transform can be written as:

$$X(\sigma+j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-(\sigma+j\Omega)t}dt=\int_{-\infty}^{\infty}\left(x(t)e^{-\sigma t}\right)e^{-j\Omega t}dt \quad (9.3)$$

Comparing (9.3) and the Fourier transform formula in (5.1):

$$X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt \quad (9.4)$$

Laplace transform of $x(t)$ is equal to the Fourier transform of $x(t)e^{-\sigma t}$.

When $\sigma=0$ or $s=j\Omega$, (9.3) and (9.4) are identical:

$$X(s)|_{s=j\Omega}=X(j\Omega)=\int_{-\infty}^{\infty}x(t)e^{-j\Omega t}dt \quad (9.5)$$

That is, Laplace transform generalizes Fourier transform, as z transform generalizes the discrete-time Fourier transform.

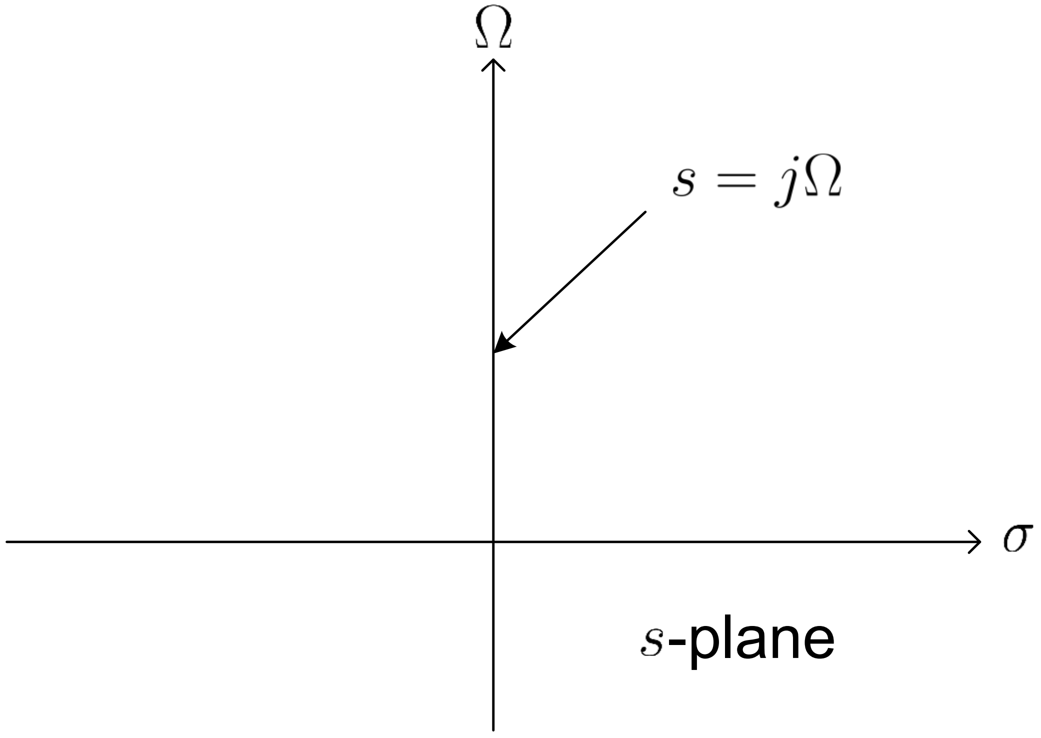

Fig. 9.1: Relationship between $X(s)$ and $X(j\Omega)$ on the s-plane

Region of Convergence (ROC)

As in z transform of discrete-time signals, ROC indicates when Laplace transform of $x(t)$ converges.

That is, if

$$|X(s)| = \left| \int_{-\infty}^{\infty} x(t) e^{-st} dt \right| \to \infty \tag{9.6}$$

then the Laplace transform does not converge at point s.

Employing $s = \sigma + j\Omega$ and $|e^{j\Omega t}| = 1$, Laplace transform exists if

$$|X(\sigma + j\Omega)| \leq \int_{-\infty}^{\infty} \left| x(t) e^{-(\sigma + j\Omega)t} \right| dt = \int_{-\infty}^{\infty} \left| x(t) e^{-\sigma t} \right| dt < \infty \tag{9.7}$$

The set of values of σ which satisfies (9.7) is called the ROC, which must be specified along with $X(s)$ in order for the Laplace transform to be complete.

Note also that if

$$|X(j\Omega)| = \left| \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \right| \rightarrow \infty \tag{9.8}$$

then the Fourier transform does not exist. While it exists if

$$|X(j\Omega)| \leq \int_{-\infty}^{\infty} \left| x(t) e^{-j\Omega t} \right| dt = \int_{-\infty}^{\infty} |x(t)| \, dt < \infty \tag{9.9}$$

Hence it is possible that the Fourier transform of $x(t)$ does not exist.

Also, the Laplace transform does not exist if there is no value of σ satisfies (9.7).

Poles and Zeros

Values of s for which $X(s)=0$ are the zeros of $X(s)$.

Values of s for which $X(s)=\pm\infty$ are the poles of $X(s)$.

Example 9.1

In many real-world applications, $X(s)$ is represented as a rational function in s:

$$X(s)=\frac{\displaystyle\sum_{k=0}^{M} b_k s^k}{\displaystyle\sum_{k=0}^{N} a_k s^k}$$

Discuss the poles and zeros of $X(s)$.

Performing factorization on $X(s)$ yields:

$$X(s)=\frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k}=\frac{b_M(s-d_1)(s-d_2)\cdots(s-d_M)}{a_N(s-c_1)(s-c_2)\cdots(s-c_N)}$$

We see that there are M nonzero zeros, namely, $d_1, d_2, \cdots, d_M$, and N nonzero poles, namely, $c_1, c_2, \cdots, c_N$.

As in z transform, we use a “$\circ$” to represent a zero and a “$\times$” to represent a pole on the s-plane.

Example 9.2

Determine the Laplace transform of $x(t) = e^{-at}u(t)$ where $u(t)$ is the unit step function and a is a real number. Determine the condition when the Fourier transform of $x(t)$ exists.

Using (9.1) and (2.22), we have

$$X(s) = \int_{-\infty}^{\infty} e^{-at}u(t)e^{-st}dt = \int_{0}^{\infty} e^{-(s+a)t}dt$$

Employing $s = \sigma + j\Omega$ yields

$$X(\sigma + j\Omega) = \int_{0}^{\infty} e^{-(\sigma+a)t}e^{-j\Omega t}dt = -\frac{1}{\sigma + a + j\Omega}e^{-(\sigma+a+j\Omega)t}\bigg|_{0}^{\infty}$$

It converges if $e^{-(\sigma+a)t}$ is bounded at $t \rightarrow \infty$, indicating that the ROC is

$$\sigma + a > 0 \text{ or } \Re\{s\} = \sigma > -a$$

For $\sigma + a > 0$, $X(s)$ is computed as

$$X(s) = -\frac{1}{\sigma + a + j\Omega} e^{-(\sigma + a + j\Omega)t} \bigg|_0^\infty = \frac{1}{(\sigma + a) + j\Omega} = \frac{1}{s + a}$$

With the ROC, the Laplace transform of $x(t) = e^{-at}u(t)$ is:

$$X(s) = \frac{1}{s + a}, \quad \Re\{s\} > -a$$

It is clear that $X(s)$ does not have zero but has a pole at $s = -a$. Using (9.5), we substitute $s = j\Omega$ to obtain

$$X(j\Omega) = \frac{1}{j\Omega + a}, \quad \Re\{s\} = 0 > -a$$

As a result, the existence condition for Fourier transform of $x(t)$ is $a > 0$. Otherwise, the Fourier transform does not exist.

In general, $X(j\Omega)$ exists if its ROC includes the imaginary axis. If $\Re\{s\} > -a$ includes $j\Omega$ axis, $a > 0$ is required.

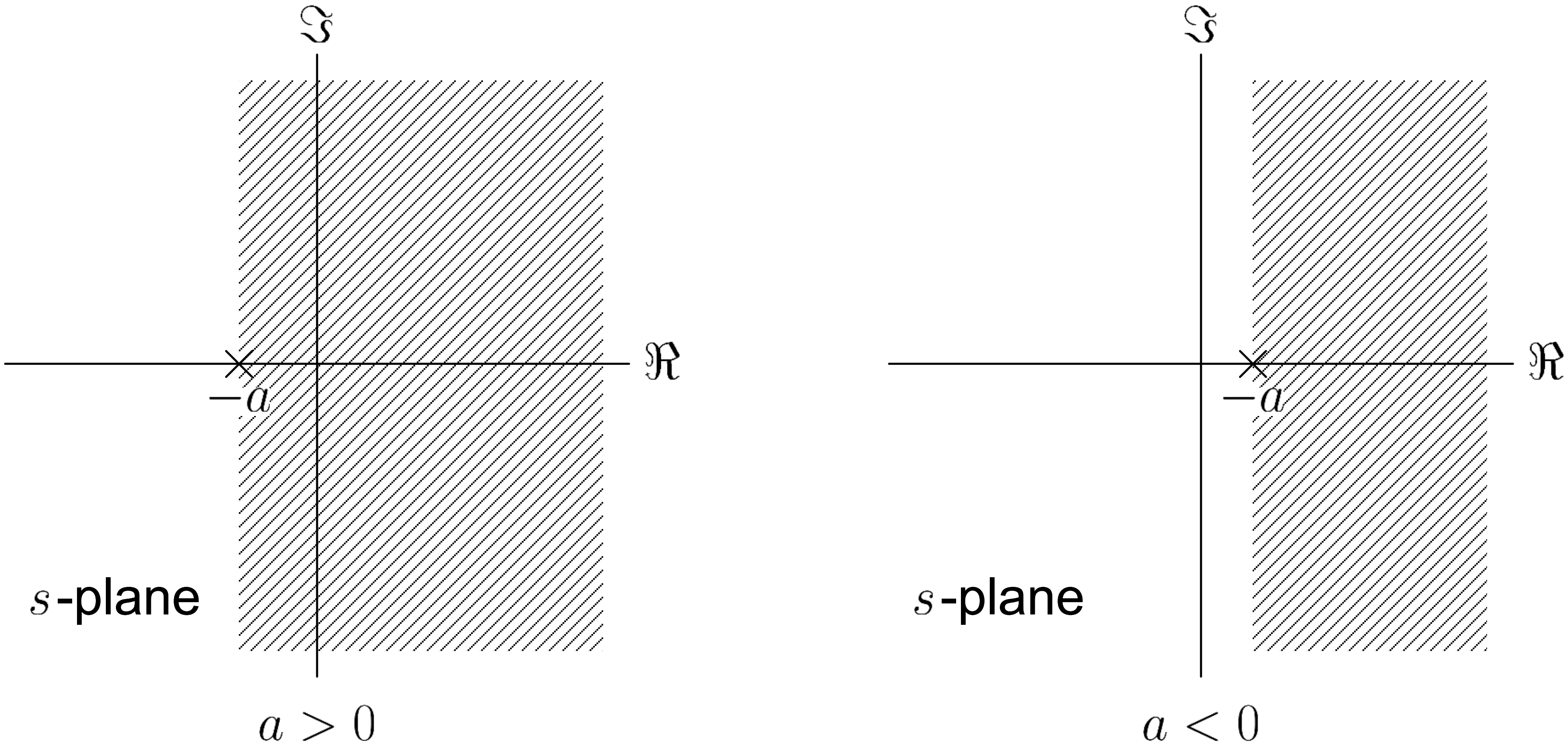

Fig. 9.2: ROCs for $a > 0$ and $a < 0$ when $x(t) = e^{-at}u(t)$

Example 9.3

Determine the Laplace transform of $x(t)=-e^{-at}u(-t)$ where a is a real number. Then determine the condition when the Fourier transform of $x(t)$ exists.

Using (9.1) and (2.22), we have

$$X(s)=\int_{-\infty}^{\infty}-e^{-at}u(-t)e^{-st}dt=-\int_{-\infty}^{0}e^{-(s+a)t}dt$$

Employing $s=\sigma+j\Omega$ yields

$$X(\sigma+j\Omega)=-\int_{-\infty}^{0}e^{-(\sigma+a)t}e^{-j\Omega t}dt=\frac{1}{\sigma+a+j\Omega}e^{-(\sigma+a+j\Omega)t}\Big|_{-\infty}^{0}$$

It converges if $e^{-(\sigma+a)t}$ is bounded at $t\rightarrow-\infty$, indicating that:

$$\sigma+a<0 \text{ or } \Re\{s\}=\sigma<-a$$

For $\sigma + a < 0$, $X(s)$ is computed as

$$X(s) = \frac{1}{\sigma + a + j\Omega} e^{-(\sigma + a + j\Omega)t} \Big|_{-\infty}^{0} = \frac{1}{(\sigma + a) + j\Omega} = \frac{1}{s + a}$$

With the ROC, the Laplace transform of $x(t) = -e^{-at}u(-t)$ is:

$$X(s) = \frac{1}{s + a}, \quad \Re\{s\} < -a$$

It is clear that $X(s)$ does not have zero but has a pole at $s = -a$. Using (9.5), we substitute $s = j\Omega$ to obtain

$$X(j\Omega) = \frac{1}{j\Omega + a}, \quad \Re\{s\} = 0 < -a$$

As a result, the existence condition for Fourier transform of $x(t)$ is $a < 0$. Otherwise, the Fourier transform does not exist.

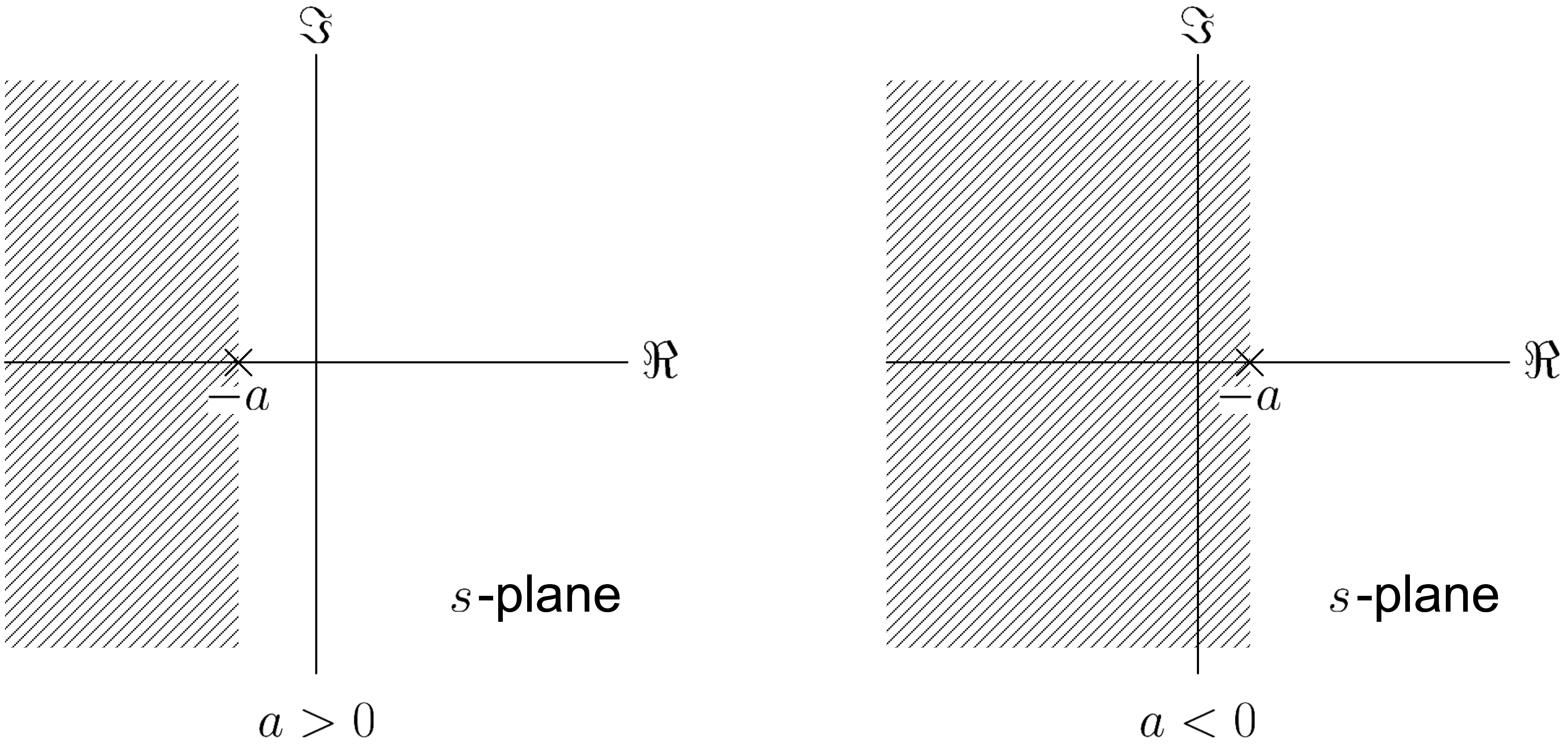

Fig. 9.3: ROCs for $a > 0$ and $a < 0$ when $x(t) = -e^{-at}u(-t)$

We also see that $X(j\Omega)$ exists if its ROC includes the imaginary axis.

Example 9.4

Determine the Laplace transform of $x(t)=e^{-at}u(t)+e^{bt}u(-t)$, assuming that a and b are real such that $b>-a$.

Employing the results in Examples 9.2 and 9.3, we have

$$\begin{aligned} X(s) &= \frac{1}{s+a}-\frac{1}{s-b}, \quad \Re\{s\}>-a,\ \Re\{s\}<b \\ &= \frac{-(a+b)}{(s+a)(s-b)}, \quad b>\Re\{s\}>-a \end{aligned}$$

Note that there is no zero while there are two poles, namely, $s=-a$ and $s=b$.

If $b<-a$, then there is no intersection between $\Re\{s\}>-a$ and $\Re\{s\}<b$, and $X(s)$ does not exist for any s.

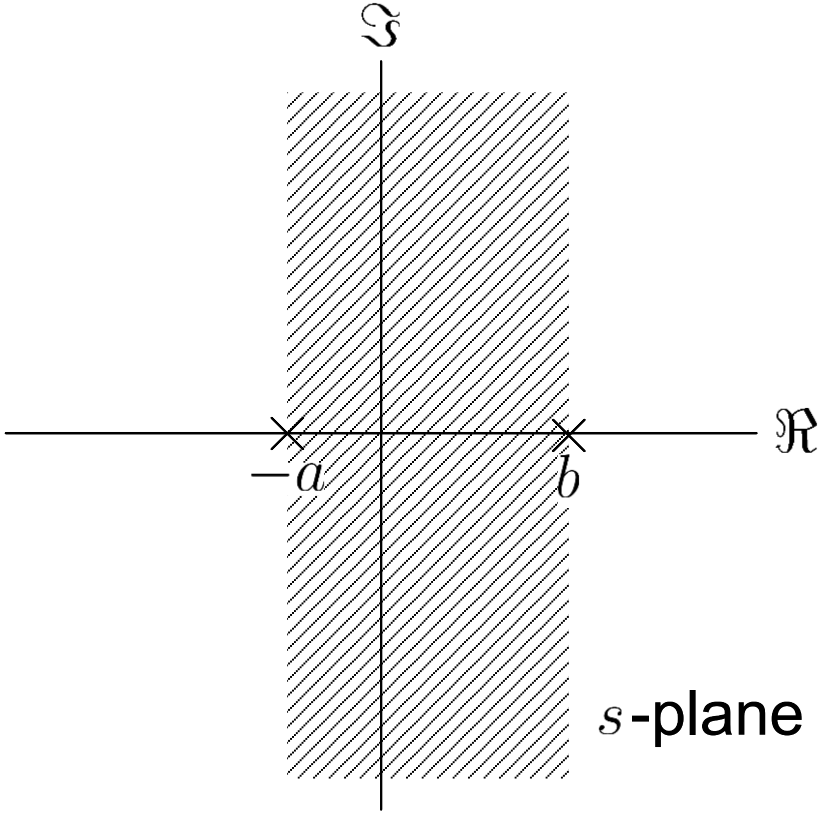

Fig. 9.4: ROC for $x(t) = e^{-at}u(t) + e^{bt}u(-t)$

Does the Fourier transform of *x(t)* exist?

Example 9.5

Determine the Laplace transform of $x(t)=\delta(t)$.

Using (9.1) and (2.19), we have

$$X(s)=\int_{-\infty}^{\infty}\delta(t)e^{-st}dt=\int_{-\infty}^{\infty}\delta(t)e^{-s\cdot 0}dt=\int_{-\infty}^{\infty}\delta(t)dt=1$$

Example 9.6

Determine the Laplace transform of $x(t)=\delta(t+1)+\delta(t-1)$.

Similar to Example 9.5, we have

$$\begin{aligned}X(s) &= \int_{-\infty}^{\infty}[\delta(t+1)+\delta(t-1)]e^{-st}dt\\ &= \int_{-\infty}^{\infty}\delta(t+1)e^{-s\cdot -1}dt+\int_{-\infty}^{\infty}\delta(t-1)e^{-s\cdot 1}dt\\ &= e^{s}+e^{-s}\end{aligned}$$

Example 9.7

Determine the Laplace transform of $x(t)=e^{-at}[u(t)-u(t-10)]$

$$\begin{aligned}
X(s) &= \int_{-\infty}^{\infty} e^{-at}[u(t)-u(t-10)]e^{-st}dt \\
&= \int_{0}^{10} e^{-(s+a)t}dt \\
&= -\frac{1}{s+a}e^{-(s+a)t}\Big|_{0}^{10} \\
&= \frac{1-e^{-10(s+a)}}{s+a}
\end{aligned}$$

What are the ROCs in Examples 9.5, 9.6 and 9.7?

Finite-Duration and Infinite-Duration Signals

Finite-duration signal: values of $x(t)$ are nonzero only for a finite time interval. If $x(t)$ is absolutely integrable, then the ROC of $X(s)$ is the entire s-plane.

Example 9.8

Given a finite-duration $x(t)$ such that:

$$x(t) = \begin{cases} \text{nonzero}, & T_1 < t < T_2 \\ 0, & \text{otherwise} \end{cases}$$

It is also absolutely integrable:

$$\int_{-\infty}^{\infty} |x(t)| dt = \int_{T_1}^{T_2} |x(t)| dt < \infty$$

Show that the ROC of $X(s)$ is the entire s-plane.

According to (9.7), $X(s)$ converges if

$$\int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt = \int_{T_1}^{T_2} \left|x(t)e^{-\sigma t}\right| dt < \infty$$

We consider three cases, namely, $\sigma = 0$, $\sigma > 0$ and $\sigma < 0$.

The convergence condition is satisfied at $\sigma = 0$ because $x(t)$ is absolutely integrable.

For $\sigma > 0$, $e^{-\sigma T_1} > e^{-\sigma t}$ for $t \in (T_1, T_2)$, and we have:

$$\int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt = \int_{T_1}^{T_2} \left|x(t)e^{-\sigma t}\right| dt < e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)|\, dt < \infty$$

because $e^{-\sigma T_1}$ is bounded and $x(t)$ is absolutely integrable.

Similarly, for $\sigma < 0$, $e^{-\sigma T_2} > e^{-\sigma t}$ for $t \in (T_1, T_2)$, and we have:

$$\int_{-\infty}^{\infty} \left|x(t)e^{-\sigma t}\right| dt = \int_{T_1}^{T_2} \left|x(t)e^{-\sigma t}\right| dt < e^{-\sigma T_2} \int_{T_1}^{T_2} \left|x(t)\right| dt < \infty$$

because $e^{-\sigma T_2}$ is bounded and $x(t)$ is absolutely integrable.

As for all values of σ, (9.7) is satisfied, hence the ROC is the entire s-plane.

If $x(t)$ is not of finite-duration, it is an infinite-duration signal:

- Right-sided: if $x(t)=0$ for $t < T_1 < \infty$ (e.g., Example 9.2 or $x(t)=e^{-at}u(t)$ with $T_1=0$; $x(t)=e^{-at}u(t-2.2)$ with $T_1=2.2$; $x(t)=e^{-at}u(t+3.3)$ with $T_1=-3.3$).
- Left-sided: if $x(t)=0$ for $t > T_2 > -\infty$ (e.g., Example 9.3 or $x(t)=e^{-at}u(-t)$ with $T_2=0$; $x(t)=e^{-at}u(-t+2.2)$ with $T_2=2.2$).
- Two-sided: neither right-sided nor left-sided (e.g., Example 9.4).

Signal	Transform	ROC
$\delta(t)$	1	All s
$\delta(t-T)$	e^{-sT}	All s
$e^{-at}u(t)$	$\frac{1}{s+a}$	$\Re\{s\} > -a$
$-e^{-at}u(-t)$	$\frac{1}{s+a}$	$\Re\{s\} < -a$
$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} > -a$
$-\frac{t^{n-1}}{(n-1)!}e^{-at}u(-t)$	$\frac{1}{(s+a)^n}$	$\Re\{s\} < -a$
$e^{-at}\cos(bt)u(t)$	$\frac{s+a}{(s+a)^2+b^2}$	$\Re\{s\} > -a$
$e^{-at}\sin(bt)u(t)$	$\frac{b}{(s+a)^2+b^2}$	$\Re\{s\} > -a$

Table 9.1: Laplace transforms for common signals

Summary of ROC Properties

P1. The ROC of $X(s)$ consists of a region parallel to the $j\Omega$-axis in the s-plane. There are four possible cases, namely, the entire region, right-half plane (region includes ∞), left-half plane (region includes $-\infty$) and single strip (region bounded by two poles).

P2. The Fourier transform of a signal $x(t)$ exists if and only if the ROC of the Laplace transform of $x(t)$ includes the $j\Omega$-axis (e.g., Examples 9.2 and 9.3).

P3: For a rational $X(s)$, its ROC cannot contain any poles (e.g., Examples 9.2 to 9.4).

P4: When $x(t)$ is of finite-duration and is absolutely integrable, the ROC is the entire s-plane (e.g., Example 9.7).

P5: When $x(t)$ is right-sided, the ROC is the right-half plane to the right of the rightmost pole (e.g., Example 9.2).

P6: When $x(t)$ is left-sided, the ROC is left-half plane to the left of the leftmost pole (e.g., Example 9.3).

P7: When $x(t)$ is two-sided, the ROC is of the form $\Re\{p_a\} > \Re\{s\} > \Re\{p_b\}$ where p_a and p_b are two poles of $X(s)$ with the successive values in real part (e.g., Example 9.4).

P8: The ROC must be a connected region.

Example 9.9

Consider a Laplace transform $X(s)$ contains three real poles, namely, a, b and c with $a < b < c$. Determine all possible ROCs.

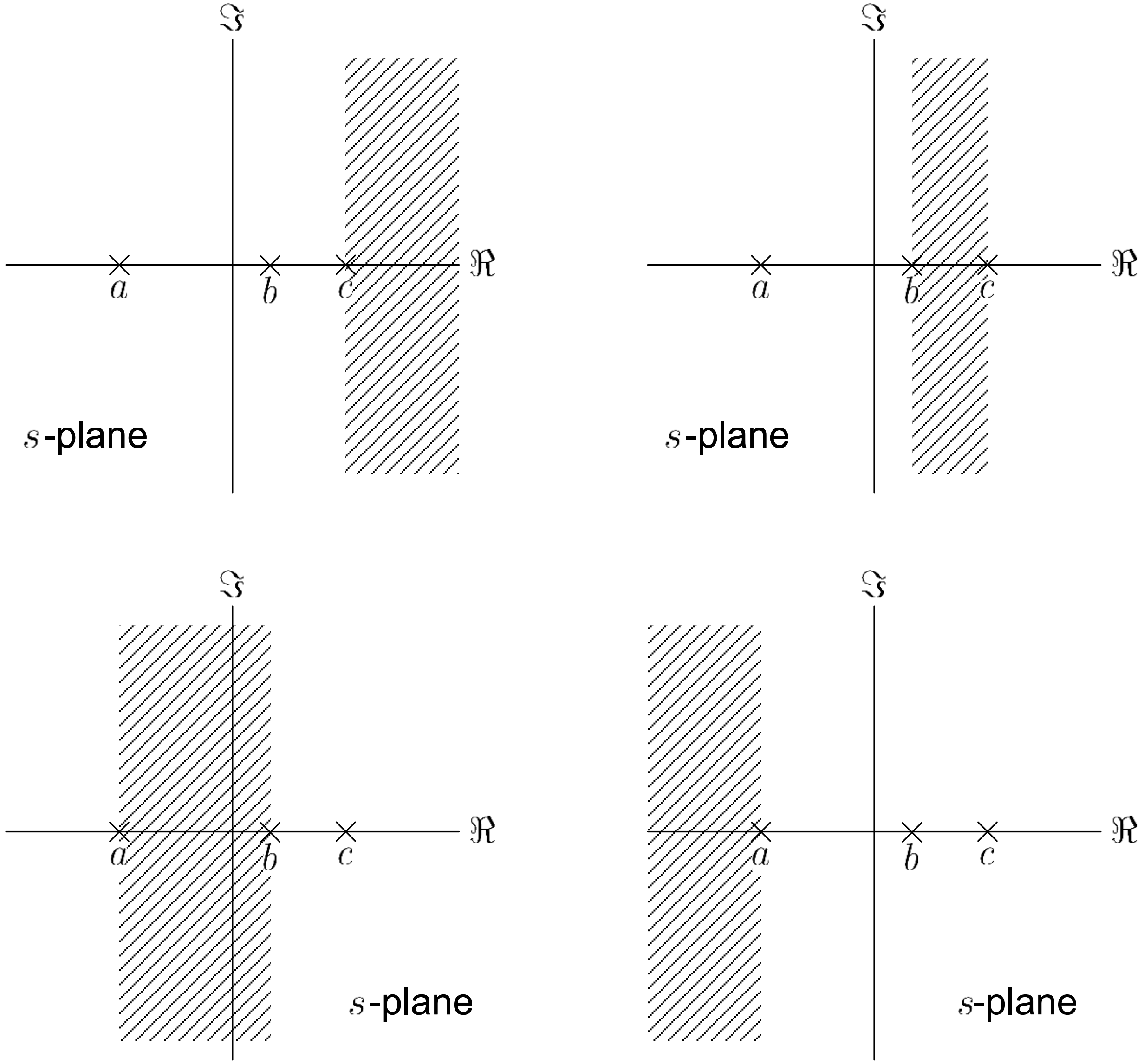

Fig.9.5: ROC possibilities for three poles

Properties of Laplace Transform

Linearity

Let $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ be two Laplace transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, we have

$$a x_1(t) + b x_2(t) \leftrightarrow a X_1(s) + b X_2(s) \tag{9.10}$$

Its ROC is denoted by $\mathcal{R}$, which includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$ where $\cap$ is the intersection operator. That is, $\mathcal{R}$ contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Example 9.10

Determine the Laplace transform of $y(t)$:

$$y(t) = x_1(t) - x_2(t)$$

where $x_1(t) = 3e^{-2t}u(t)$ and $x_2(t) = 2e^{-t}u(t)$. Find also the pole and zero locations.

From Table 9.1, we have:

$$e^{-2t}u(t) \leftrightarrow \frac{1}{s+2}, \quad \Re\{s\} > -2$$

and

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

According to the linearity property, the Laplace transform of $y(t)$ is

$$Y(s) = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2}, \quad \Re\{s\} > -1$$

There are two poles, namely -2 and -1 and there is one zero at 1.

Example 9.11

Determine the ROC of the Laplace transform of $y(t)$ which is expressed as:

$$y(t) = x_1(t) - x_2(t)$$

The Laplace transforms of $x_1(t)$ and $x_2(t)$ are:

$$X_1(s) = \frac{1}{s+1},\ \Re\{s\} > -1 \quad \text{and} \quad X_2(s) = \frac{1}{(s+1)(s+2)},\ \Re\{s\} > -1$$

We have:

$$Y(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}$$

We can deduce that the ROC of $y(t)$ is $\Re\{s\} > -2$, which contains the intersection of the ROCs of $X_1(s)$ and $X_2(s)$ which is $\Re\{s\} > -1$. Note also that the pole at $s = -1$ is cancelled by the zero at $s = -1$.

Time Shifting

A time-shift of t_0 in $x(t)$ causes a multiplication of e^{-st_0} in $X(s)$

$$x(t) \leftrightarrow X(s) \Rightarrow x(t-t_0) \leftrightarrow e^{-st_0}X(s) \qquad (9.11)$$

The ROC for $x(t-t_0)$ is identical to that of $X(s)$.

Example 9.12

Find the Laplace transform of $x(t)$ which has the form of:

$$x(t) = e^{-at}u(t-10)$$

Employing the time shifting property with $t=10$ and:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

we easily obtain

$$e^{-10a} \cdot e^{-a(t-10)}u(t-10) \leftrightarrow e^{-10a} \cdot e^{-10s}\frac{1}{s+a} = \frac{e^{-10(s+a)}}{s+a}, \quad \Re\{s\} > -a$$

Multiplication by an Exponential Signal

If we multiply $x(t)$ by $e^{s_0 t}$ in the time domain, the variable s will be changed to $s - s_0$ in the Laplace transform domain:

$$x(t) \leftrightarrow X(s) \Rightarrow e^{s_0 t}x(t) \leftrightarrow X(s - s_0) \tag{9.12}$$

If the ROC for $x(t)$ is $\mathcal{R}$, then the ROC for $e^{s_0 t}x(t)$ is $\mathcal{R} + \Re\{s_0\}$, that is, shifted by $\Re\{s_0\}$. Note that if $X(s)$ has a pole (zero) at $s = a$, then $X(s - s_0)$ has a pole (zero) at $s = a + s_0$.

Example 9.13

With the use of the following Laplace transform pair:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

Find the Laplace transform of $x(t)$ which has the form of:

$$e^{-at}\cos(bt)u(t)$$

Noting that $\cos(bt) = (e^{jbt} + e^{-jbt})/2$, $x(t)$ can be written as:

$$x(t) = \frac{1}{2}e^{(-a+jb)t}u(t) + \frac{1}{2}e^{(-a-jb)t}u(t)$$

By means of the property of (9.12) with the substitution of $s_0 = jb$ and $s_0 = -jb$, we obtain:

$$\frac{1}{2}e^{jbt}[e^{-at}u(t)] \leftrightarrow \frac{1}{2}\frac{1}{(s-jb)+a}, \quad \Re\{s\} > -a$$

and

$$\frac{1}{2}e^{-jbt}[e^{-at}u(t)] \leftrightarrow \frac{1}{2}\frac{1}{(s+jb)+a}, \quad \Re\{s\} > -a$$

By means of the linearity property, it follows that

$$X(s) = \frac{1}{2}\frac{1}{(s-jb)+a} + \frac{1}{2}\frac{1}{(s+jb)+a} = \frac{s+a}{(s+a)^2+b^2}, \quad \Re\{s\} > -a$$

which agrees with Table 9.1.

Differentiation in s Domain

Differentiating $X(s)$ with respect to s corresponds to multiplying $x(t)$ by $-t$ in the time domain:

$$x(t) \leftrightarrow X(s) \Rightarrow -tx(t) \leftrightarrow \frac{dX(s)}{ds} \qquad (9.13)$$

The ROC for $tx(t)$ is identical to that of $X(s)$.

Example 9.14

Determine the Laplace transform of $x(t) = te^{-at}u(t)$.

We start with using:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

and

$$\frac{d}{ds}\left(\frac{1}{s+a}\right) = -\frac{1}{(s+a)^2}$$

Applying (9.13), we obtain:

$$te^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^2}, \quad \Re\{s\} > -a$$

Further differentiation yields:

$$\tfrac{t^2}{2}e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^3}, \quad \Re\{s\} > -a$$

The result can be generalized as:

$$\frac{t^{n-1}}{(n-1)!}e^{-at}u(t) \leftrightarrow \frac{1}{(s+a)^n}, \quad \Re\{s\} > -a$$

which agrees with Table 9.1.

Conjugation

The Laplace transform pair for $x^*(t)$ is:

$$x(t) \leftrightarrow X(s) \Rightarrow x^*(t) \leftrightarrow X^*(s^*) \qquad (9.14)$$

The ROC for $x^*(t)$ is identical to that of $X(s)$.

Hence when $x(t)$ is real-valued, $X(s) = X^*(s^*)$.

Time Reversal

The Laplace transform pair for $x(-t)$ is:

$$x(t) \leftrightarrow X(s) \Rightarrow x(-t) \leftrightarrow X(-s) \qquad (9.15)$$

The ROC will be reversed as well. For example, if the ROC for $x(t)$ is $\Re\{s\} > -a$, then the ROC for $x(-t)$ is $\Re\{s\} < a$.

Example 9.15

Determine the Laplace transform of $x(t)=e^{at}u(-t)$.

We start with using:

$$e^{-at}u(t) \leftrightarrow \frac{1}{s+a}, \quad \Re\{s\} > -a$$

Applying (9.15) yields

$$e^{at}u(-t) \leftrightarrow \frac{1}{-s+a} = -\frac{1}{s-a}, \quad \Re\{s\} < a$$

Convolution

Let $x_1(t) \leftrightarrow X_1(s)$ and $x_2(t) \leftrightarrow X_2(s)$ be two Laplace transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively. Then we have:

$$x_1(t) \otimes x_2(t) \leftrightarrow X_1(s)X_2(s) \tag{9.16}$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$. The proof is similar to (5.23).

Differentation in Time Domain

Differentiating $x(t)$ with respect to t corresponds to multiplying $X(s)$ by s in the s-domain:

$$x(t) \leftrightarrow X(s) \Rightarrow \frac{dx(t)}{dt} \leftrightarrow sX(s) \qquad (9.17)$$

Its ROC includes the ROC for $x(t)$.

Repeated application of (9.17) yields the general form:

$$\frac{d^k x(t)}{dt^k} \leftrightarrow s^k X(s) \qquad (9.18)$$

Example 9.16

Use the Laplace transform of $u(t)$ to determine the Laplace transform of $x(t)=\delta(t)$.

According to (2.24):

$$\delta(t)=\frac{du(t)}{dt}$$

Substituting $a=0$ into Example 9.2 or Table 9.1, we have:

$$u(t)\leftrightarrow\frac{1}{s},\quad \Re\{s\}>0$$

Employing (9.17) and (2.24) yields

$$\delta(t) \leftrightarrow s \cdot \frac{1}{s} = 1$$

where the ROC is the entire s-plane.

Note that the result can be easily extended to the derivative of $\delta(t)$. For example,

$$\frac{d\delta(t)}{dt} \leftrightarrow s \cdot 1 = s$$

Extension using (9.18) yields:

$$\frac{d^n\delta(t)}{dt^n} \leftrightarrow s^n$$

Integration

On the other hand, if we perform integration on $x(t)$, this corresponds to dividing $X(s)$ by s in the s-domain:

$$x(t) \leftrightarrow X(s) \Rightarrow \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow \frac{1}{s}X(s) \qquad (9.19)$$

If the ROC for $x(t)$ is $\mathcal{R}$, then the ROC for $\int_{-\infty}^{t} x(\tau)d\tau$ includes $\mathcal{R} \cap \{\Re\{s\} > 0\}$.

Example 9.17

Prove (9.19), that is, the integration property of Laplace transform.

We first notice that

$$x(t) \otimes u(t) = \int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} x(\tau)d\tau$$

because $u(t-\tau)=1$ only for $\tau \in (-\infty, t)$.

Applying the convolution property of (9.16) and noting from Example 9.16 that

$$u(t) \leftrightarrow \frac{1}{s}, \quad \Re\{s\} > 0$$

We then have:

$$x(t) \otimes u(t) = \int_{-\infty}^{t} x(\tau)d\tau \leftrightarrow X(s) \cdot \frac{1}{s}$$

where the ROC includes the intersection of ROC of $X(s)$ and $\Re\{s\} > 0$.

Example 9.18

Determine the Laplace transform of $x(t) = u(t) \otimes u(t)$.

From Example 9.17, we know that

$$u(t) \otimes u(t) = \int_{-\infty}^{t} u(\tau) d\tau$$

Employing (9.19) and

$$u(t) \leftrightarrow \frac{1}{s}, \quad \Re\{s\} > 0$$

We then have:

$$u(t) \otimes u(t) \leftrightarrow \frac{1}{s} \cdot \frac{1}{s} = \frac{1}{s^2}, \quad \Re\{s\} > 0$$

Alternatively, this can be easily obtained using (9.16). Note that its generalization is:

$$\underbrace{u(t) \otimes \cdots \otimes u(t)}_{n \text{ times}} = \frac{1}{s^n}, \quad \Re\{s\} > 0$$

Causality and Stability Investigation with ROC

Suppose $h(t)$ is the impulse response of a continuous-time linear time-invariant (LTI) system. Recall (3.18), which is the causality condition:

$$h(t) = 0, \quad t < 0 \tag{9.20}$$

If the system is causal and $h(t)$ is of infinite duration, the ROC must be the right-half plane, i.e., the region of the right of the rightmost pole, indicating it is right-sided. Note that causality implies right-half plane ROC but the converse may not be true.

Nevertheless, if $H(s)$ is rational and its ROC is the right-half plane, then the system must be causal.

Example 9.19

Discuss the causality of the two LTI systems with impulse responses $h_1(t)$ and $h_2(t)$. Their Laplace transforms are:

$$H_1(s) = \frac{1}{s+1}, \quad \Re\{s\} > -1, \qquad H_2(s) = \frac{e^s}{s+1}, \quad \Re\{s\} > -1$$

For $H_1(s)$, we use Table 9.1 or Example 9.2 to obtain:

$$h_1(t) = e^{-t}u(t)$$

which corresponds to a causal system. We can also know its causality because $H_1(s)$ is rational and its ROC is the right-half plane.

On the other hand, using the time-shifting property and the above result, we have:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1 \Rightarrow e^{-(t+1)}u(t+1) \leftrightarrow \frac{e^s}{s+1}, \quad \Re\{s\} > -1$$

That is,

$$h_2(t) = e^{-(t+1)}u(t+1)$$

which corresponds to a non-causal system. This also aligns with the above discussion because $H_2(s)$ is not rational although its ROC is also right-half plane.

Recall the stability condition in (3.20):

$$\int_{-\infty}^{\infty} |h(t)| dt < \infty \tag{9.21}$$

(9.21) corresponds to the existence condition of the Fourier transform of $h(t)$. According to P2, this means that the ROC of $H(s)$ includes the $j\Omega$-axis.

That is, a LTI system is stable if and only if the ROC of $H(s)$ includes the $j\Omega$-axis.

Example 9.20

Discuss the causality and stability of a LTI system with impulse response $h(t)$. The Laplace transform of $h(t)$ is:

$$H(s) = \frac{3}{s+1} + \frac{2}{s-2}$$

As the ROC of $H(s)$ is not specified, we investigate all possible cases, i.e., $\Re\{s\} < -1$, $-1 < \Re\{s\} < 2$ and $\Re\{s\} > 2$.

For $\Re\{s\} < -1$, we use Table 9.1 to obtain:

$$-e^{-t}u(-t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} < -1$$

and

$$-e^{2t}u(-t) \leftrightarrow \frac{1}{s-2}, \quad \Re\{s\} < 2$$

where both ROCs agree with $\Re\{s\} < -1$. Combining the results yields:

$$h(t) = -\left[3e^{-t} + 2e^{2t}\right]u(-t)$$

Because of $u(-t)$ and e^{-t} is approaching unbounded as $t \to -\infty$, this system is non-causal and unstable.

Similarly we obtain for $-1 < \Re\{s\} < 2$:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

and

$$-e^{2t}u(-t) \leftrightarrow \frac{1}{s-2}, \quad \Re\{s\} < 2$$

Combining the results yields:

$$h(t) = 3e^{-t}u(t) - 2e^{2t}u(-t)$$

Due to $u(-t)$, the system is not causal. While e^{-t} is absolutely integrable in $t \in (0, \infty)$ and e^{2t} is absolutely integrable in $t \in (-\infty, 0)$, the system is stable.

Finally, for $\Re\{s\} > 2$, we use:

$$e^{-t}u(t) \leftrightarrow \frac{1}{s+1}, \quad \Re\{s\} > -1$$

and

$$e^{2t}u(t) \leftrightarrow \frac{1}{s-2}, \quad \Re\{s\} > 2$$

Combining the results yields:

$$h(t) = 3e^{-t}u(t) + 2e^{2t}u(t)$$

This system is causal but not stable due to $e^{2t}u(t)$.

To summarize, a causal system with rational $H(s)$ is stable if and only if all of the poles of $H(s)$ lies in the left-half of the s -plane, i.e., all of the poles have negative real parts.

Inverse Laplace Transform

Inverse Laplace transform corresponds to finding $x(t)$ given $X(s)$ and its ROC.

The Laplace transform and inverse Laplace transform are one-to-one mapping provided that the ROC is given:

$$x(t) \leftrightarrow X(s) \tag{9.22}$$

There are 3 commonly used techniques to perform the inverse Laplace transform. They are

1. Inspection
2. Partial Fraction Expansion
3. Contour Integration

Inspection

When we are familiar with certain transform pairs, we can do the inverse Laplace transform by inspection.

Example 9.21

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{s-1}{s+1}, \quad \Re\{s\} > -1$$

Reorganizing $X(s)$ as:

$$X(s) = \frac{s+1-2}{s+1} = 1 - \frac{2}{s+1}, \quad \Re\{s\} > -1$$

Using Table 9.1 and linearity property, we get:

$$x(t) = \delta(t) - 2e^{-t}u(t)$$

Partial Fraction Expansion

The technique is identical to that in inverse z transform but now we consider that $X(s)$ is a rational function in s:

$$X(s) = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \tag{9.23}$$

To obtain the partial fraction expansion from (9.23), the first step is to determine N nonzero poles, $c_1, c_2, \cdots, c_N$.

There are 4 cases to be considered:

Case 1: $M < N$ and all poles are of first order

$X(s)$ can be decomposed as:

$$X(s) = \sum_{k=1}^{N} \frac{A_k}{s - c_k} \tag{9.24}$$

For each first-order term of $A_k / (s - c_k)$, its inverse Laplace transform can be easily obtained by inspection.

The A_k can be computed as:

$$A_k = (s - c_k) X(s)\big|_{s=c_k} \tag{9.25}$$

Case 2: $M \geq N$ and all poles are of first order

In this case, $X(s)$ can be expressed as:

$$X(s) = \sum_{l=0}^{M-N} B_l s^l + \sum_{k=1}^{N} \frac{A_k}{s - c_k} \tag{9.26}$$

- B_l are obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator.

- A_k can be obtained using (9.25).

Case 3: $M < N$ with multiple-order pole(s)

Assuming that $X(s)$ has a r-order pole at $s = c_i$ with $r \geq 2$, then $X(s)$ can be decomposed as:

$$X(s) = \sum_{k=1, k \neq i}^{N} \frac{A_k}{s - c_k} + \sum_{m=1}^{r} \frac{C_m}{(s - c_i)^m} \tag{9.27}$$

- When there are two or more multiple-order poles, we include a component like the second term for each corresponding pole

- A_k can be computed according to (9.25)
- C_m can be calculated from:

$$C_m = \frac{1}{(r-m)!} \cdot \frac{d^{r-m}}{ds^{r-m}} \left[(s-c_i)^r X(s) \right] \bigg|_{s=c_i} \qquad (9.28)$$

Case 4: $M \geq N$ with multiple-order pole(s)

Assuming that $X(s)$ has a r-order pole at $s = c_i$ with $r \geq 2$, then $X(s)$ can be decomposed as:

$$X(s) = \sum_{l=0}^{M-N} B_l s^l + \sum_{k=1, k\neq i}^{N} \frac{A_k}{s - c_k} + \sum_{m=1}^{r} \frac{C_m}{(s-c_i)^m} \qquad (9.29)$$

The A_k, B_l and C_m can be calculated as in Cases 1, 2 and 3.

Example 9.22

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{2s^2+9s-11}{(s+1)(s-2)(s+3)}, \quad \Re\{s\} > 2$$

We can express $X(s)$ as:

$$X(s) = \frac{A_1}{s+1} + \frac{A_2}{s-2} + \frac{A_3}{s+3}$$

Employing (9.25), A_1, A_2 and A_3 are:

$$A_1 = \left.\frac{2s^2+9s-11}{(s-2)(s+3)}\right|_{s=-1} = 3$$

$$A_2 = \left.\frac{2s^2+9s-11}{(s+1)(s+3)}\right|_{s=2} = 1$$

and

$$A_3 = \left.\frac{2s^2+9s-11}{(s+1)(s-2)}\right|_{s=-3} = -2$$

Together with the ROC of $\Re\{s\} > 2$, we obtain:

$$x(t) = 3e^{-t}u(t) + e^{2t}u(t) - 2e^{-3t}u(t)$$

Example 9.23

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{2s^3+9s^2+11s+2}{s^2+4s+3}, \quad \Re\{s\} > -1$$

First we perform long division to obtain:

$$X(s) = 2s + 1 + \frac{s-1}{s^2+4s+3}$$

The last term can be further decomposed as:

$$\frac{s-1}{(s+1)(s+3)} = \frac{A_1}{s+1} + \frac{A_2}{s+3}$$

Employing (9.25), A_1 and A_2 are:

$$A_1 = \left.\frac{s-1}{s+3}\right|_{s=-1} = -1$$

and

$$A_2 = \left.\frac{s-1}{s+1}\right|_{s=-3} = 2$$

Together with the ROC of $\Re\{s\} > -1$, we obtain:

$$x(t) = 2\frac{d\delta(t)}{dt} + \delta(t) - e^{-t}u(t) + 2e^{-3t}u(t)$$

Example 9.24

Find $x(t)$ if its Laplace transform has the form of:

$$X(s) = \frac{s+2}{(s+1)^2(s+3)}, \quad \Re\{s\} > -1$$

Accordingly to (9.27), we can express $X(s)$ as:

$$X(s) = \frac{A_1}{s+3} + \frac{C_1}{s+1} + \frac{C_2}{(s+1)^2}$$

Employing (9.25), A_1 is:

$$A_1 = \left.\frac{s+2}{(s+1)^2}\right|_{s=-3} = -\frac{1}{4}$$

Applying (9.28), C_1 and C_2 are:

$$C_1 = \frac{1}{(2-1)!} \cdot \frac{d}{ds}\left.\left(\frac{s+2}{s+3}\right)\right|_{s=-1} = \left.\frac{s+3-(s+2)}{(s+3)^2}\right|_{s=-1} = \frac{1}{4}$$

and

$$C_2 = \frac{1}{(2-2)!} \cdot \left.\frac{s+2}{s+3}\right|_{s=-1} = \frac{1}{2}$$

Together with the ROC of $\Re\{s\} > -1$, we obtain:

$$x(t) = -0.25e^{-3t}u(t) + 0.25e^{-t}u(t) + 0.5te^{-t}u(t)$$

Transfer Function of Linear Time-Invariant System

A LTI system can be characterized by the transfer function, which is a Laplace transform expression.

Starting with the differential equation in (3.25) which describes the continuous-time LTI system:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k} \qquad (9.30)$$

Applying Laplace transform on (9.30) with the use of the linearity property and (9.18), we have:

$$Y(s) \sum_{k=0}^{N} a_k s^k = X(s) \sum_{k=0}^{M} b_k s^k \qquad (9.31)$$

The transfer function, denoted by $H(s)$, is defined as:

$$H(s) = \frac{Y(s)}{X(s)} = \frac{\sum_{k=0}^{M} b_k s^k}{\sum_{k=0}^{N} a_k s^k} \tag{9.32}$$

The system impulse response $h(t)$ is given by the inverse Laplace transform of $H(s)$ with an appropriate ROC, that is, $h(t) \leftrightarrow H(s)$, such that $y(t) = x(t) \otimes h(t)$. This suggests that we can first take the Laplace transforms of $x(t)$ and $h(t)$, then multiply $X(s)$ by $H(s)$, and finally perform the inverse Laplace transform of $X(s)H(s)$ to obtain $y(t)$.

Comparing with (5.30), we see that the system frequency response can be obtained as $H(s)|_{s=j\Omega} = H(j\Omega)$ if it exists.

Example 9.25

Determine the transfer function for a LTI system whose input $x(t)$ and output $y(t)$ are related by:

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

Taking Laplace transform on the both sides with the use of the linearity and differentiation properties, $H(s)$ is:

$$Y(s)(s+3) = X(s) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1}{s+3}$$

Note that there are two ROC possibilities, namely, $\Re\{s\} > -3$ and $\Re\{s\} < -3$, and we cannot uniquely determine $h(t)$. However, if it is known that the system is causal, $h(t)$ can be uniquely found because the ROC should be $\Re\{s\} > -3$.

Example 9.26

Find the differential equation corresponding to a continuous-time LTI system whose transfer function is given by

$$H(s)=\frac{s+3}{(s+1)(s+2)}$$

Let $H(s)=Y(s)/X(s)$. Performing cross-multiplication and inverse Laplace transform, we obtain:

$$\begin{aligned}
&(s+1)(s+2)Y(s)=(s+3)X(s)\\
&\Rightarrow \left(s^2+3s+2\right)Y(s)=(s+3)X(s)\\
&\Rightarrow \frac{d^2y(t)}{dt^2}+3\frac{dy(t)}{dt}+2y(t)=\frac{dx(t)}{dt}+3x(t)
\end{aligned}$$

Examples 9.25 and 9.26 imply the equivalence between the differential equation and transfer function.

Example 9.27

Compute the impulse response $h(t)$ for a LTI system which is characterized by the following equation:

$$y(t) = x(t) - x(t-1)$$

Applying Laplace transform on the input-ouput equation using the linearity and time shifting properties, $H(s)$ is:

$$Y(s) = X(s)\left(1 - e^{-s}\right) \Rightarrow H(s) = \frac{Y(s)}{X(s)} = 1 - e^{-s}$$

From Table 9.1, there is only one ROC possibility, i.e., entire s-plane. Taking the inverse Laplace transform on $H(s)$ yields:

$$h(t) = \delta(t) - \delta(t-1)$$

which agrees with Example 3.10.

Example 9.28

Compute the impulse response $h(t)$ for a LTI system which is characterized by the following equation:

$$y(t) = \frac{1}{10}\int_0^{10} x(t-\tau)d\tau$$

Noting that

$$\begin{aligned}\frac{1}{10}\int_0^{10} x(t-\tau)d\tau &= 0.1\int_{-\infty}^{\infty}[u(\tau)-u(\tau-10)]x(t-\tau)d\tau \\ &= 0.1\left[x(t)\otimes u(t) - x(t)\otimes u(t-10)\right]\end{aligned}$$

Taking the Laplace transform on the input-output relationship and using convolution as well as time-shifting properties, we get:

$$Y(s) = 0.1\left[X(s)\cdot\frac{1}{s} - X(s)\cdot\frac{e^{-10s}}{s}\right] \Rightarrow H(s) = \frac{Y(s)}{X(s)} = \frac{1-e^{-10s}}{10s}$$

As the system is causal because $y(t)$ does not depend on future input, we can deduce that the ROC of $H(s)$ is $\Re\{s\} > 0$.

Finally, taking the inverse Laplace transform on $H(s)$ yields:

$$h(t) = 0.1[u(t) - u(t-10)]$$

which agrees with Example 3.11.

<u>Example 9.29</u>

Compute the output $y(t)$ if the input is $x(t) = e^{-at}u(t)$ with $a > 0$ and the LTI system impulse response is $h(t) = u(t)$.

The Laplace transforms of $x(t)$ and $h(t)$ are

$$X(s) = \frac{1}{s+a}, \quad \Re\{s\} > -a$$

and

$$H(s) = \frac{1}{s}, \quad \Re\{s\} > 0$$

As a result, we have:

$$Y(s) = X(s)H(s) = \frac{1}{a}\left[\frac{1}{s} - \frac{1}{s+a}\right], \quad \Re\{s\} > 0$$

Taking the inverse Laplace transform of $Y(s)$ with the ROC of $\Re\{s\} > 0$ yields:

$$y(t) = \frac{1}{a}\left(1 - e^{-at}\right)u(t)$$

which agrees with Example 3.16.