

FIR Filter Design

Chapter Intended Learning Outcomes:

- (i) Understanding of the characteristics of linear-phase finite impulse response (FIR) filters
- (ii) Ability to design linear-phase FIR filters according to predefined specifications using the window and frequency sampling methods

Steps in Digital Filter Design

1. Specification Determination

The first step is to obtain the filter specifications or requirements which are determined by the applications.

Suppose we have observations $r[n] = s[n] + q[n]$ where $s[n]$ is the signal of interest and $q[n]$ is the additive noise. The $s[n]$ only has low-frequency components such that $S(e^{j\omega}) = 0$ for $\omega > \omega_c$ while $q[n]$ is of wideband such that $Q(e^{j\omega}) \neq 0$ for the whole frequency range.

If our task is to find $s[n]$ from $r[n]$, we can use a lowpass filter with cutoff frequency of ω_c to obtain a noise-reduced version of $s[n]$.

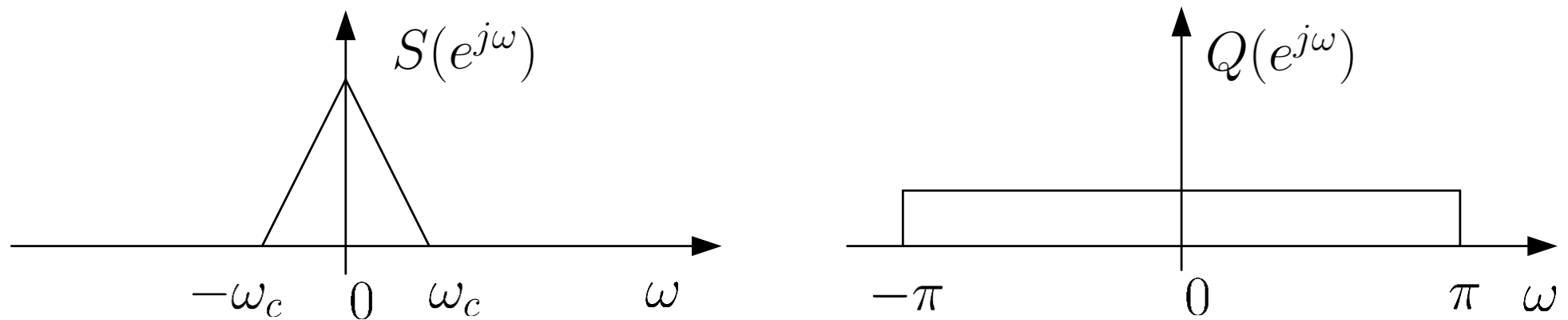


Fig.10.1: Spectra of signal and noise

The filter specification can be described by discrete-time Fourier transform (DTFT) $H(e^{j\omega})$:

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & -\pi < \omega < -\omega_c, \quad \omega_c < \omega < \pi \end{cases} \quad (10.1)$$

which specifies both the **magnitude** and **phase**

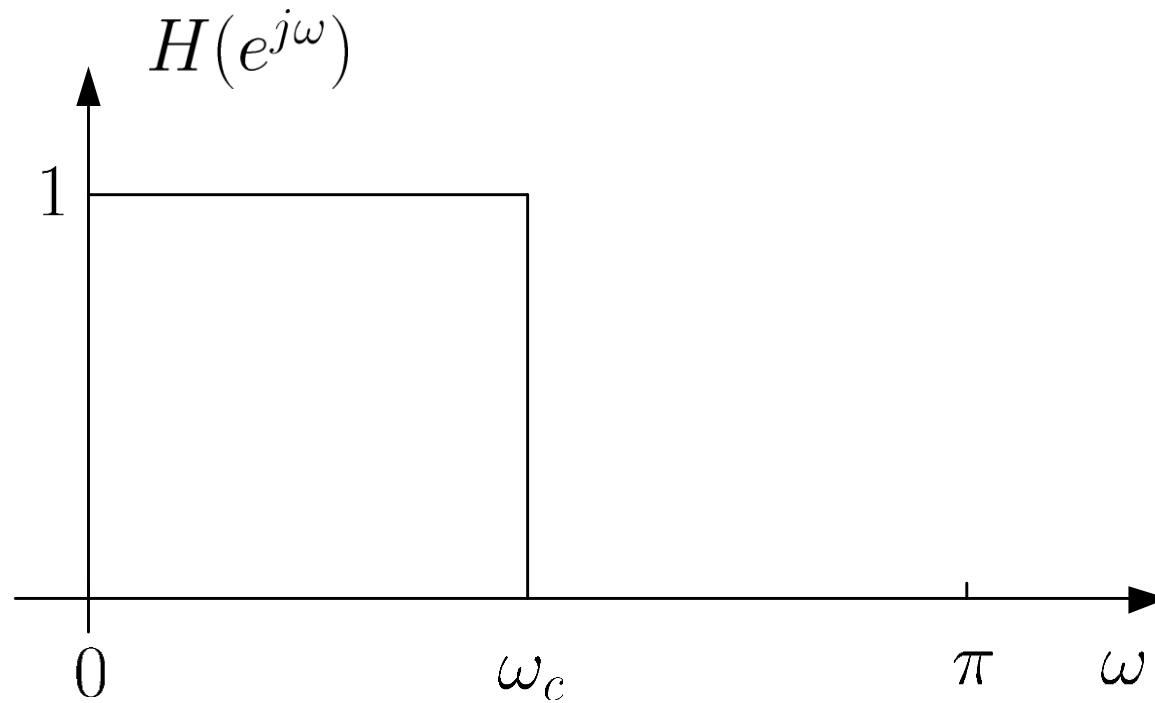


Fig.10.2: Ideal lowpass filter

- unity gain for the whole range of $\omega \in (0, \omega_c)$
- complete suppression for $\omega \in (\omega_c, \pi)$
- step change in frequency response at $\omega = \omega_c$

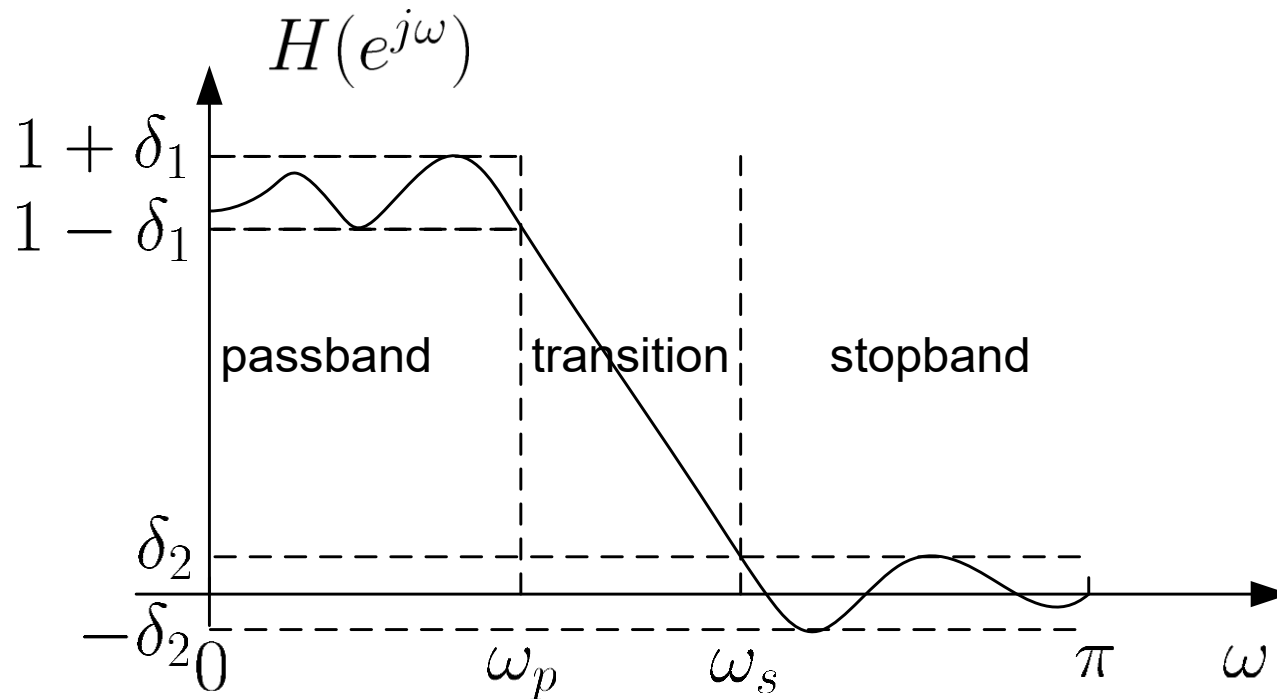


Fig.10.3: Practical $H(e^{j\omega})$

- **passband** corresponds to $\omega \in (0, \omega_p)$ where ω_p is the passband frequency and δ_1 is the passband ripple or tolerance which is the maximum allowable deviation from unity in this band

- **stopband** corresponds to $\omega \in (\omega_s, \pi)$ where ω_s is the stopband frequency and δ_2 is the stopband ripple or tolerance which is the maximum allowable deviation from zero in this band
- **transition band** corresponds to $\omega_c \in (\omega_p, \omega_s)$ where there are no restrictions on $H(e^{j\omega})$ in this band

2. Filter Response Calculation

We then use digital signal processing techniques to obtain a filter description in terms of transfer function $H(z)$ or impulse response $h[n]$ that fulfills the given specifications

3. Implementation

When $H(z)$ or $h[n]$ are known, the filter can then be realized in hardware or software according to a given structure

Advantages of FIR Filter

Transfer function of a causal FIR filter $H(z)$ with length N is:

$$H(z) = \sum_{n=0}^{N-1} h[n]z^{-n} \quad (10.2)$$

where $h[n]$ is the finite-duration impulse response

- Phase response can be exactly **linear** which results in **computation reduction** and **zero phase distortion**. Note that when there is phase distortion, different frequency components of a signal will undergo different delays in the filtering process
- It is always **stable** because $\sum_{n=0}^{N-1} |h[n]| < \infty$ for finite N
- Less sensitive to finite word-length effects

- **Efficient implementation** via digital signal processors with multiply-and-add (MAC) instruction
- Existence of optimality theorem for FIR filter design

Linear-Phase FIR Filter

The phase response of a linear-phase filter is a **linear** function of the frequency ω :

$$\angle H(e^{j\omega}) = \phi - \tau\omega, \quad -\pi < \omega < \pi \quad (10.3)$$

where ϕ equals 0 or $\pi/2$ and τ is a constant which is function of the filter length.

Note that (10.3) is slightly different from that in previous chapters where the phase response complements with the magnitude response $|H(e^{j\omega})|$.

Here, the phase response is related to $H(e^{j\omega})$ by

$$H(e^{j\omega}) = H_r(e^{j\omega})e^{j\angle H(e^{j\omega})} \quad (10.4)$$

where $H_r(e^{j\omega})$ is **real** and is called the **amplitude response**.

Comparing

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{j\angle H(e^{j\omega})} \quad (10.5)$$

where $|H(e^{j\omega})|$ is nonnegative, we see that when $H_r(e^{j\omega})$ is positive, the phase responses in (10.4) and (10.5) are identical but there is a phase difference of π for a negative $H_r(e^{j\omega})$.

Example 10.1

Determine the amplitude response and the corresponding phase response for a FIR filter with impulse response $h[n]$:

$$h[n] = u[n] - u[n - N]$$

Plot the spectra for $N = 10$. Compare the results with the magnitude response and the corresponding phase response.

Using Example 6.2:

$$H(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} \cdot e^{-j\omega(N-1)/2}$$

Hence the amplitude response is

$$H_r(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)}$$

and the corresponding phase response is

$$\angle H(e^{j\omega}) = -\frac{N-1}{2}\omega$$

On the other hand, the magnitude response and the corresponding phase response are:

$$|H(e^{j\omega})| = \left| \frac{\sin(\omega N/2)}{\sin(\omega/2)} \right|$$

and

$$\angle(H(e^{j\omega})) = -\frac{\omega(N-1)}{2} + \angle\left(\frac{\sin(\omega N/2)}{\sin(\omega/2)}\right)$$

The MATLAB program is provided as `ex10_1.m`.

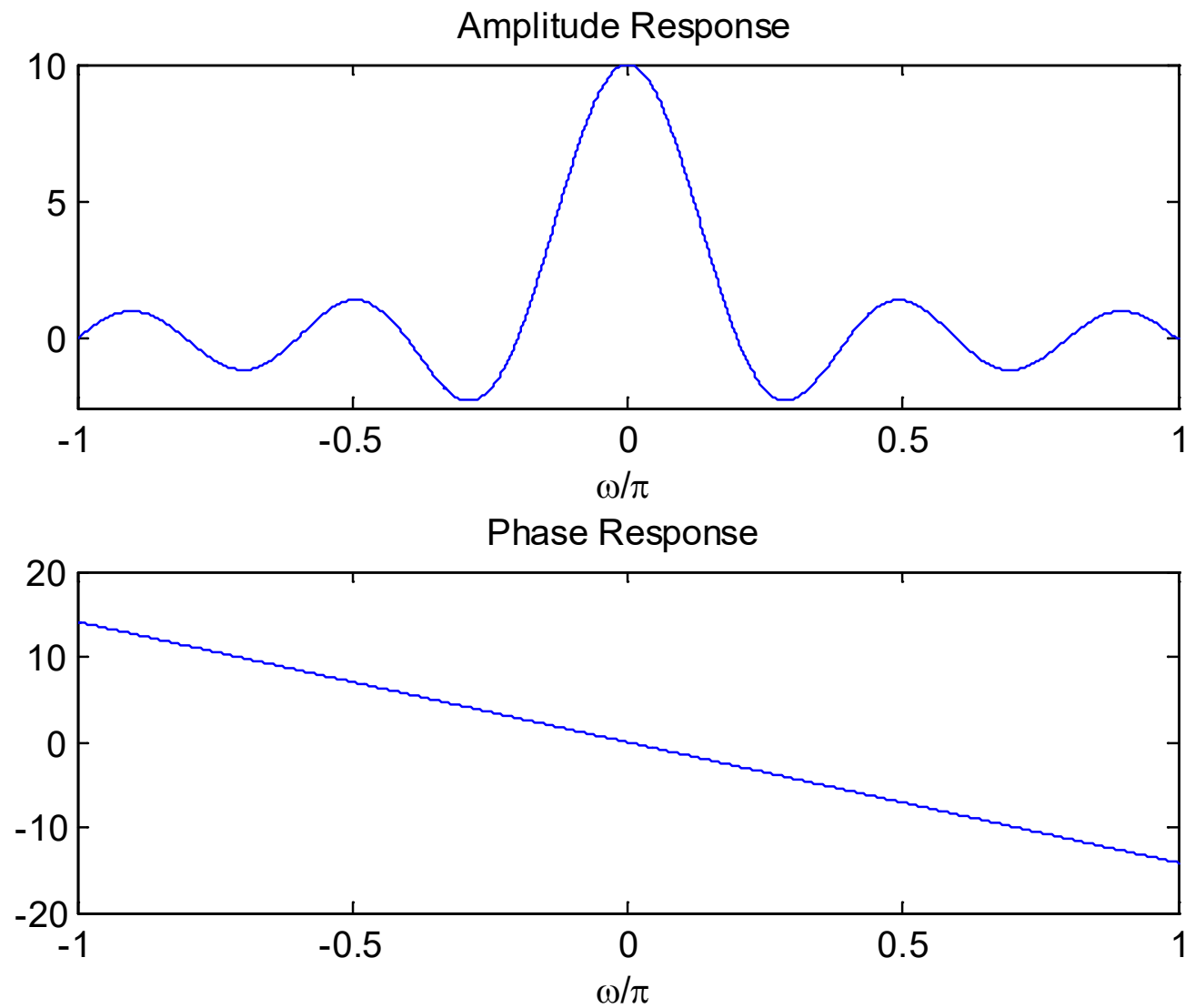


Fig.10.4: Amplitude and phase spectra

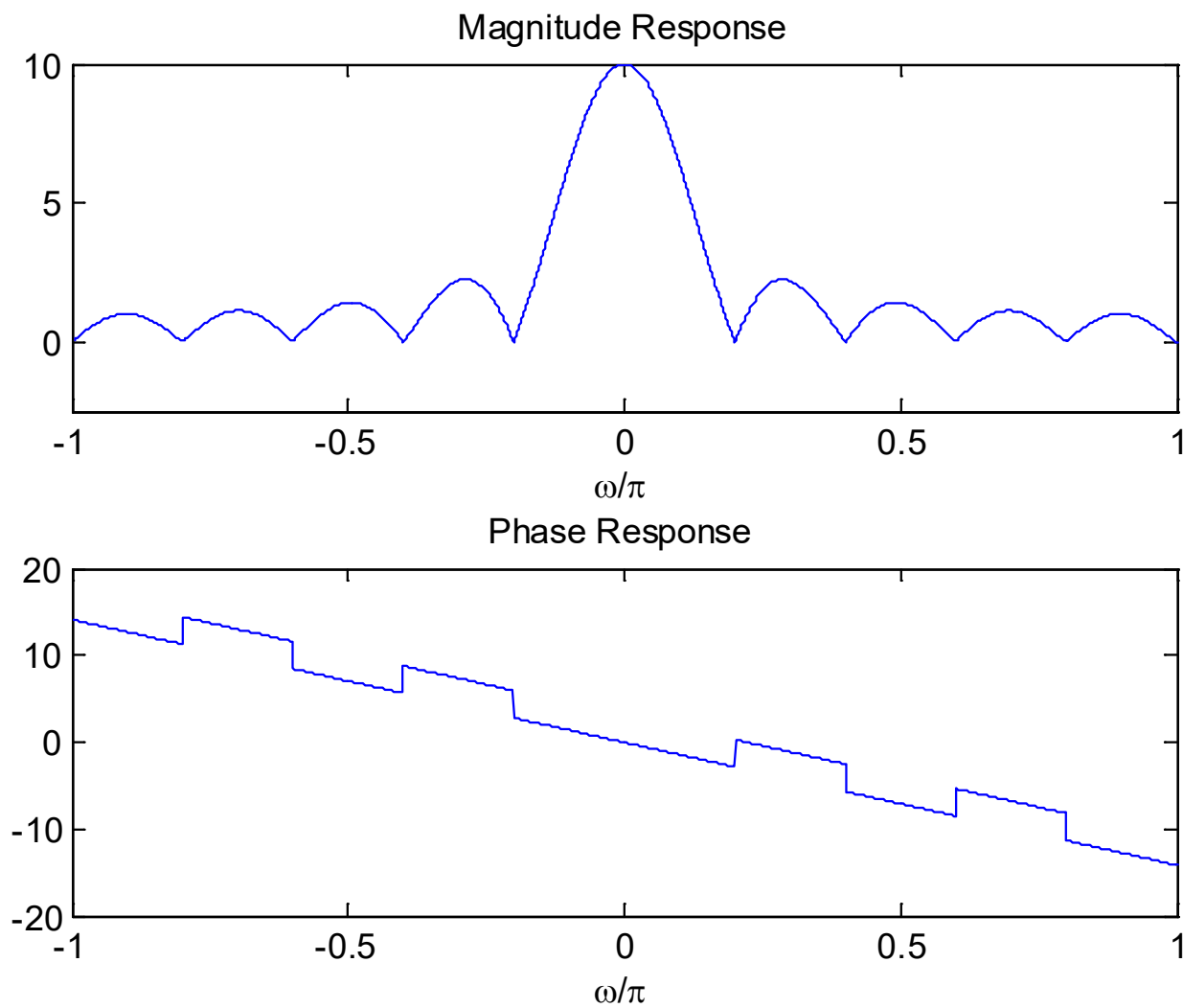


Fig.10.5: Magnitude and phase spectra

For a causal linear-phase FIR filter, the impulse response $h[n]$ is either **symmetric** or **anti-symmetric**:

$$h[n] = h[N - 1 - n], \quad 0 \leq n \leq N - 1 \quad (10.6)$$

or

$$h[n] = -h[N - 1 - n], \quad 0 \leq n \leq N - 1 \quad (10.7)$$

For example, when N is even in (10.7):

$$h[0] = -h[N - 1], h[1] = -h[N - 2], \dots, h[N/2 - 1] = -h[N/2]$$

$$\begin{aligned} y[n] &= \sum_{k=0}^{N-1} h[k]x[n - k] \\ &= h[0]x[n] + h[1]x[n - 1] + \dots + h[N/2 - 1]x[n - N/2 + 1] \\ &\quad + h[N/2]x[n - N/2] + \dots + h[N - 1]x[n - N + 1] \\ &= h[0](x[n] - x[n - N + 1]) + h[1](x[n - 1] - x[n - N + 2]) + \dots \\ &\quad + h[N/2 - 1](x[n - N/2 + 1] - x[n - N/2]) \end{aligned} \quad (10.8)$$

How many multipliers and additions are needed?

Example 10.2

Draw the block diagram using the direct form with minimum number of multiplications for the FIR filter whose impulse response is:

$$h[n] = h[4 - n], \quad 0 \leq n \leq 4$$

Following (10.8), we obtain:

$$\begin{aligned} y[n] &= \sum_{k=0}^4 h[k]x[n - k] \\ &= h[0] (x[n] + x[n - 4]) + h[1] (x[n - 1] + x[n - 3]) + h[2]x[n - 2] \end{aligned}$$

The number of multiplications is reduced from 5 to 3 while the number of additions remains unchanged, which is 4.

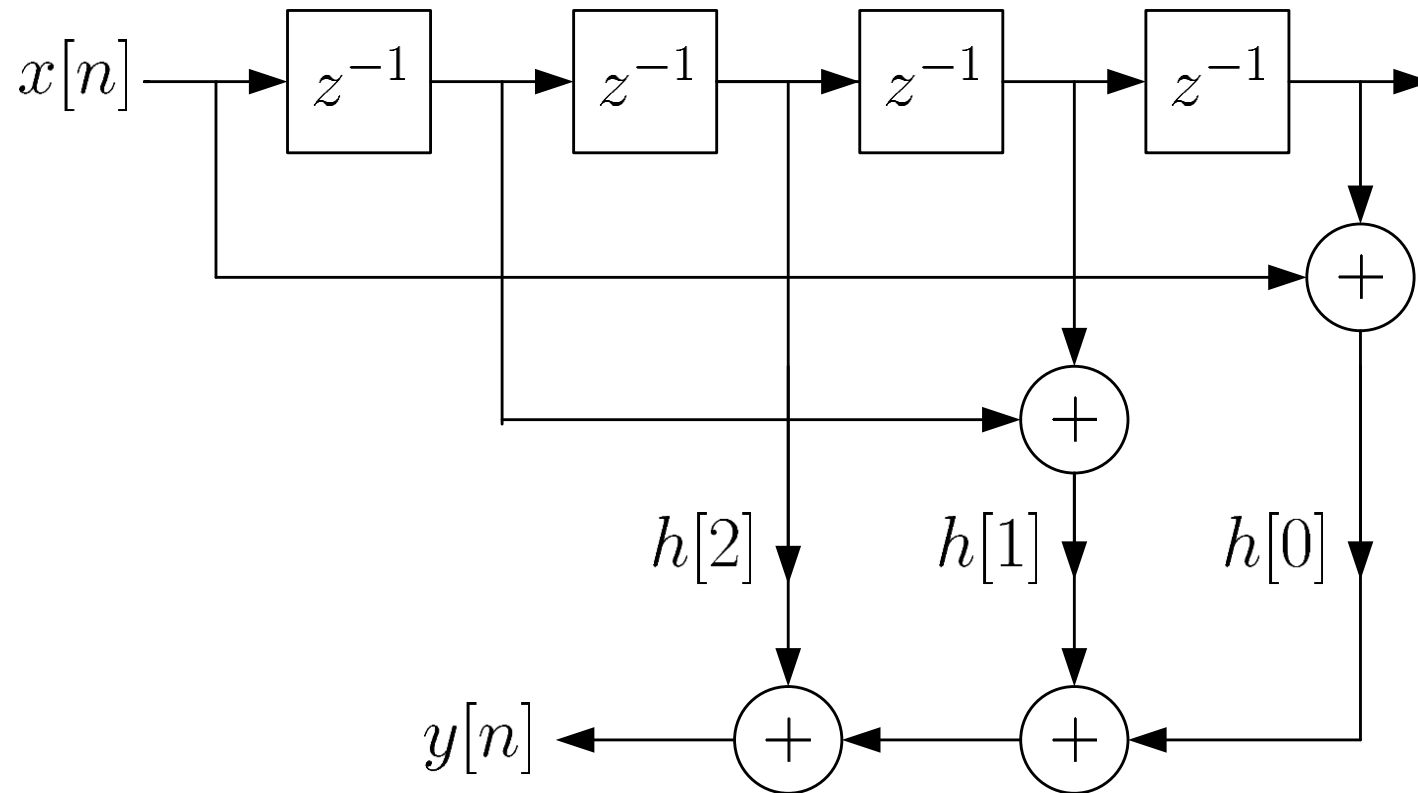


Fig.10.6: Block diagram for symmetric impulse response

There are four types of linear-phase FIR filters:

1. **Symmetric** Impulse Response with **Odd** N

Taking the DTFT of $h[n]$ yields

$$H(e^{j\omega}) = \left(2 \sum_{n=0}^{\frac{N-1}{2}-1} h[n] \cos \left(\left(\frac{N-1}{2} - n \right) \omega \right) + h[(N-1)/2] \right) e^{-j\omega(N-1)/2} \quad (10.9)$$

where $\phi = 0$ and $\tau = (N-1)/2$ is an integer.

Example 10.3

The impulse response $h[n]$ of a causal FIR filter satisfies

$$h[n] = h[N - 1 - n], \quad 0 \leq n \leq N - 1$$

with N is odd. Show that

$$\angle H(e^{j\omega}) = -\frac{N-1}{2}\omega$$

According to (10.6) with odd N , we have

$$\begin{aligned} h[0] &= h[N-1] \\ h[1] &= h[N-2] \\ &\dots\dots\dots \\ h[(N-1)/2-1] &= h[(N-1)/2+1] \end{aligned}$$

Hence

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{N-1} h[n]e^{-j\omega n} \\ &= h[0] + \dots + h[(N-1)/2-1]e^{-j\omega((N-1)/2-1)} + h[(N-1)/2]e^{-j\omega(N-1)/2} \\ &\quad + h[(N-1)/2+1]e^{-j\omega((N-1)/2+1)} + \dots + h[N-1]e^{-j\omega(N-1)} \\ &= e^{-j\omega(N-1)/2} \left(h[0]e^{j\omega(N-1)/2} + h[N-1]e^{-j\omega(N-1)/2} \right. \\ &\quad \left. + \dots + h[(N-1)/2-1]e^{j\omega} + h[(N-1)/2+1]e^{-j\omega} + h[(N-1)/2] \right) \\ &= e^{-j\omega(N-1)/2} \left(2h[0] \cos(\omega(N-1)/2) + \dots + 2h[(N-1)/2-1] \cos(\omega) \right. \\ &\quad \left. + h[(N-1)/2] \right) \\ &= e^{-j\omega(N-1)/2} \left(2 \sum_{n=0}^{\frac{N-1}{2}-1} h[n] \cos \left(\left(\frac{N-1}{2} - n \right) \omega \right) + h[(N-1)/2] \right) \end{aligned}$$

which validates (10.9) with $\angle H(e^{j\omega}) = -\omega(N-1)/2$.

2. **Symmetric** Impulse Response with **Even** N

The DTFT of $h[n]$ is:

$$H(e^{j\omega}) = 2 \left(\sum_{n=0}^{\frac{N}{2}-1} h[n] \cos \left(\left(\frac{N-1}{2} - n \right) \omega \right) \right) e^{-j\omega(N-1)/2} \quad (10.10)$$

where $\phi = 0$ and $\tau = (N-1)/2$ is not an integer.

3. **Anti-symmetric** Impulse Response with **Odd** N

The DTFT of $h[n]$ is:

$$H(e^{j\omega}) = 2 \left(\sum_{n=0}^{\frac{N-1}{2}-1} h[n] \sin \left(\left(\frac{N-1}{2} - n \right) \omega \right) \right) e^{j(\pi/2 - \omega(N-1)/2)} \quad (10.11)$$

where $\phi = \pi/2$ and $\tau = (N - 1)/2$ is an integer

Note that $h[(N - 1)/2] = 0$ due to anti-symmetric property

4. **Anti-symmetric** Impulse Response with **Even** N

Taking the DTFT of $h[n]$ yields:

$$H(e^{j\omega}) = 2 \left(\sum_{n=0}^{\frac{N}{2}-1} h[n] \sin \left(\left(\frac{N-1}{2} - n \right) \omega \right) \right) e^{j(\pi/2 - \omega(N-1)/2)} \quad (10.12)$$

where $\phi = \pi/2$ and $\tau = (N - 1)/2$ is not an integer.

Example 10.4

Consider an input sequence $x[n]$ of length 400 such that $x[n] = 0$ for $0 \leq n \leq 99$ and $200 \leq n \leq 299$, and it is a sinusoid with frequencies 0.03π and 0.08π for $100 \leq n \leq 199$ and $300 \leq n \leq 399$, respectively.

Examine the filter output $y[n]$ with the following two FIR systems:

- (a) $h[n] = \{2, -0.9, -0.72, -0.58, -0.46, -0.37\}$ with $h[0] = 2$
- (b) $h[n] = \{0.6, 0.9, -1.2, 0.9, 0.6\}$ with $h[0] = 0.6$

Are they linear phase filters?

The MATLAB program for this example is provided as `ex10_4.m`.

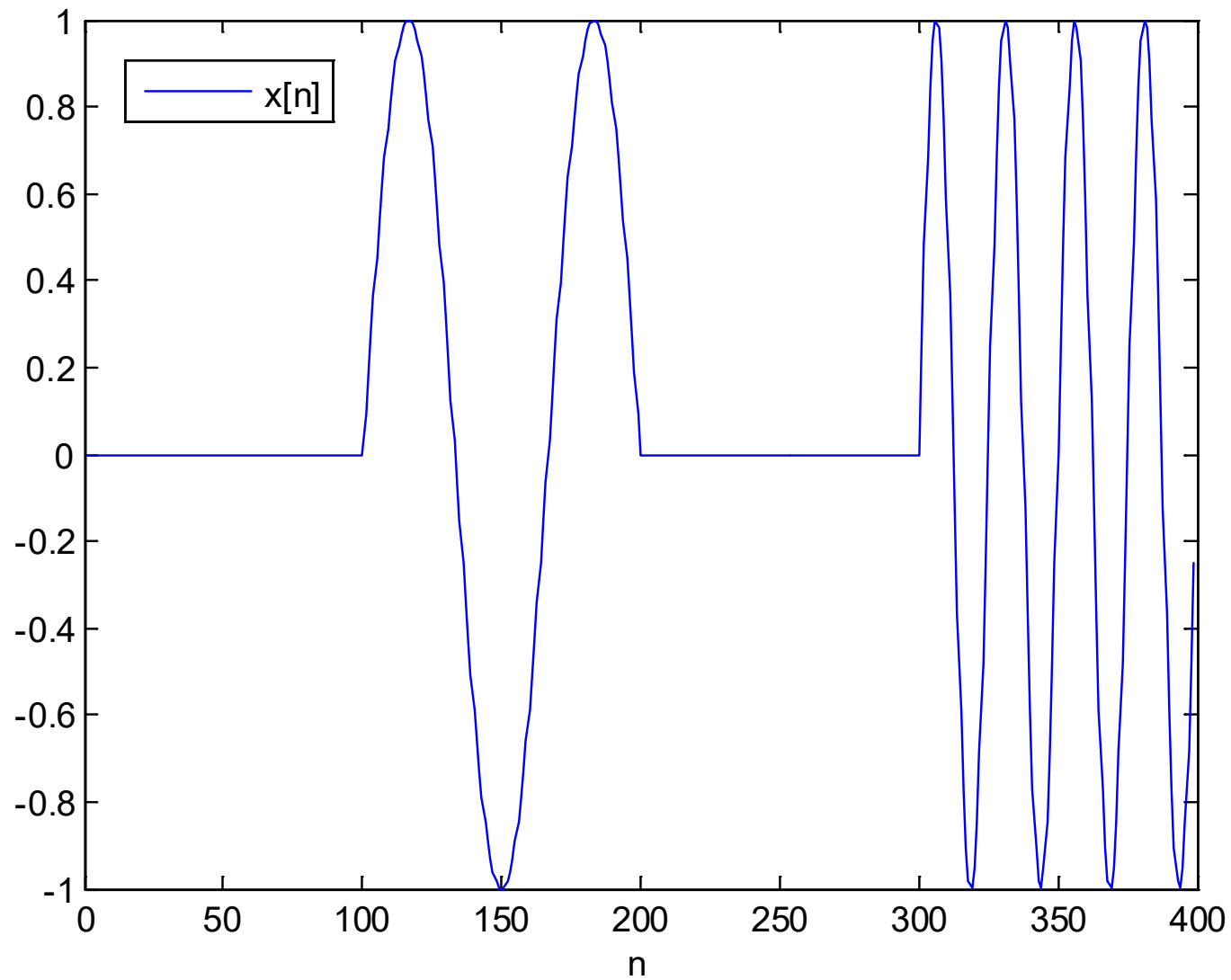


Fig.10.7: Pulsed sinusoid with two frequencies

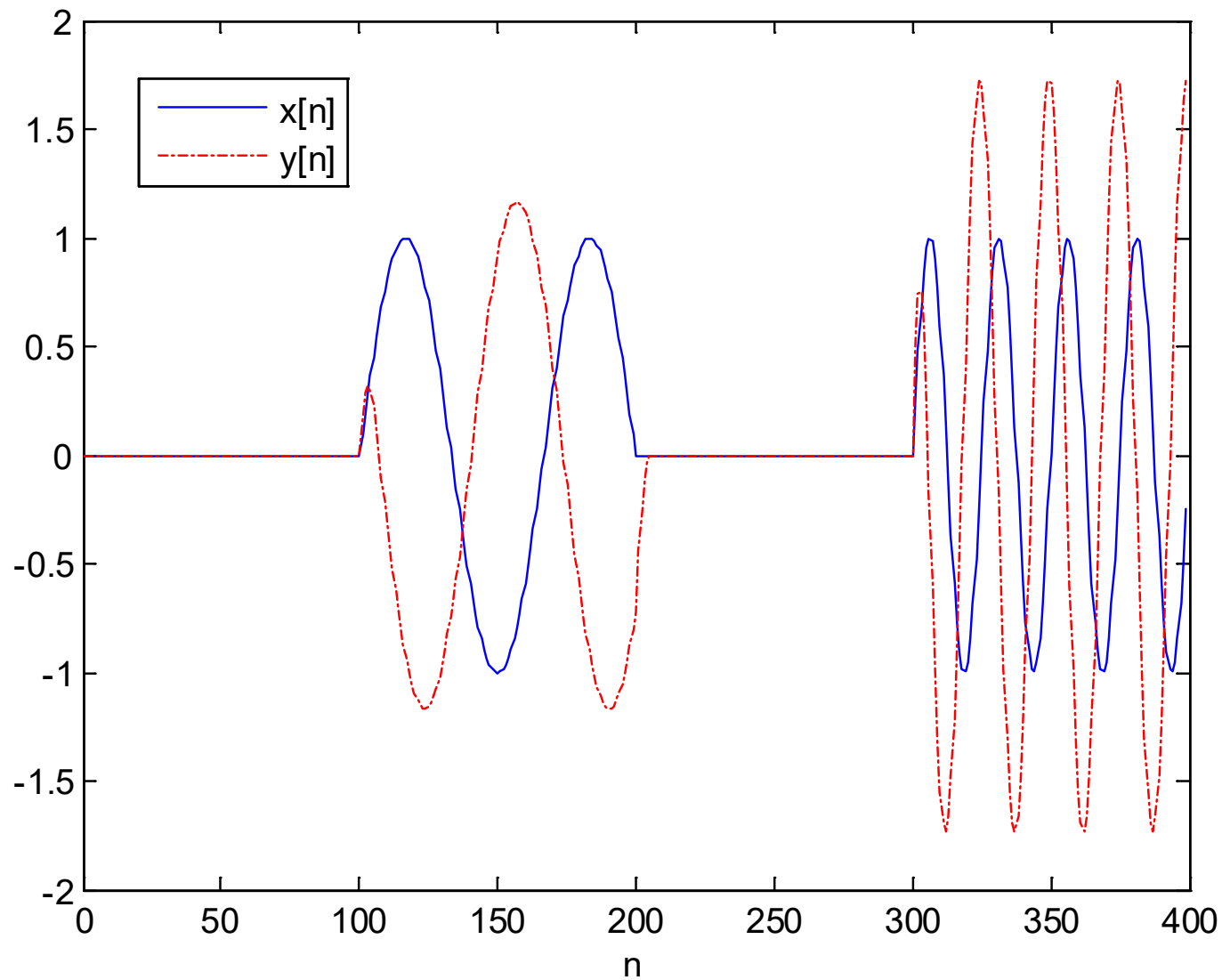


Fig.10.8: Filter output with nonlinear-phase filter

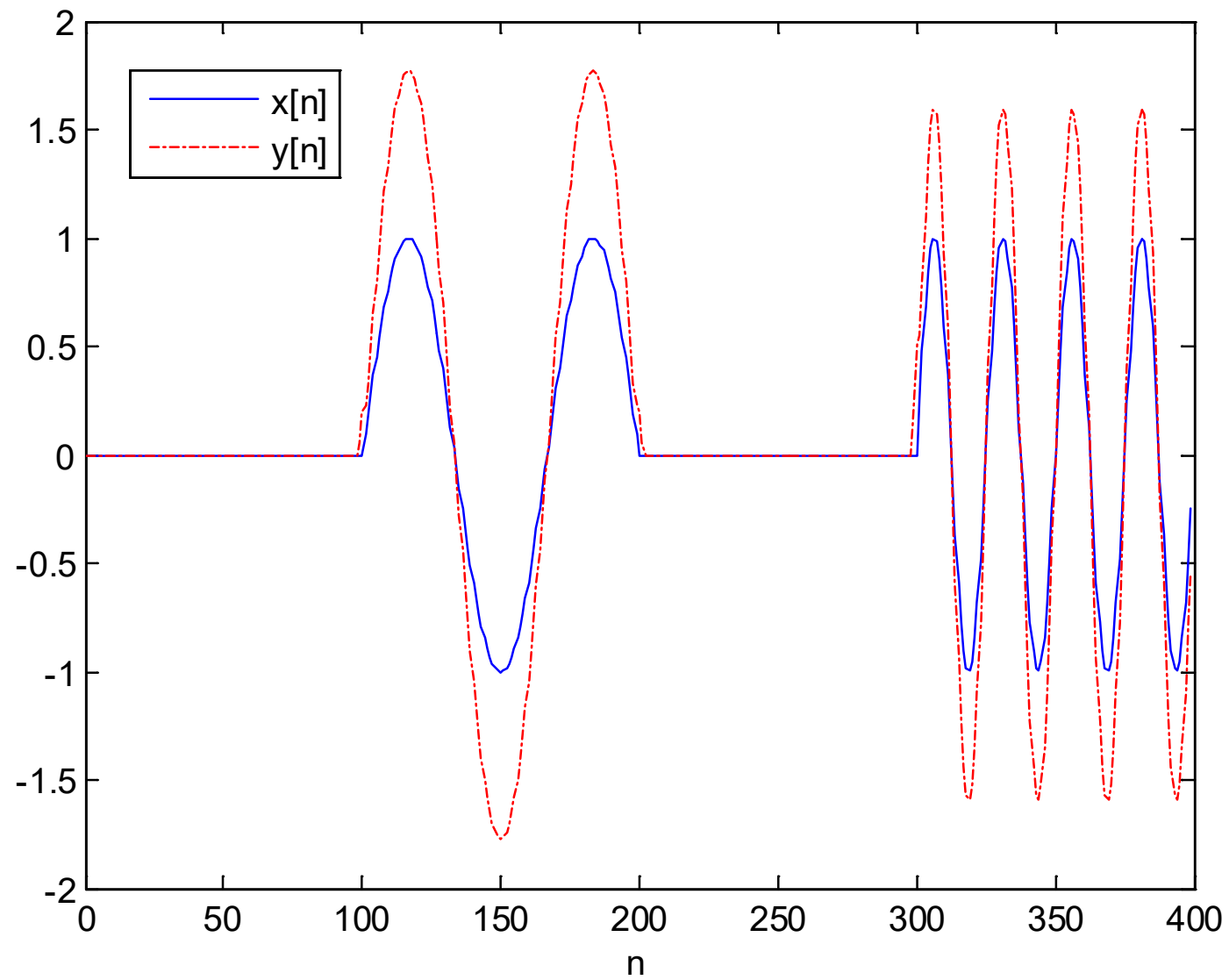


Fig.10.9: Filter output with linear-phase filter

Example 10.5

The impulse response of a causal linear-phase FIR $h[n]$ is:

$$h[n] = \{1/5, -1/4, 1/3, -1/2, 1, 0, -1, 1/2, -1/3, 1/4, -1/5\}$$

where $h[0] = 1/5$. Determine $H(e^{j\omega})$. Plot the amplitude and phase responses and then deduce the function of the filter. Find the expected filter output $y[n]$ when passing an input sequence of the form $x[n] = \sin(0.6n)$ through the filter.

As $N = 11$ is odd and $h[n]$ is anti-symmetric, we apply (10.11):

$$H(e^{j\omega}) = \left(2 \sin(\omega) - \sin(2\omega) + \frac{2}{3} \sin(3\omega) - \frac{1}{2} \sin(4\omega) + \frac{2}{5} \sin(5\omega) \right) e^{j(\pi/2 - 5\omega)}$$

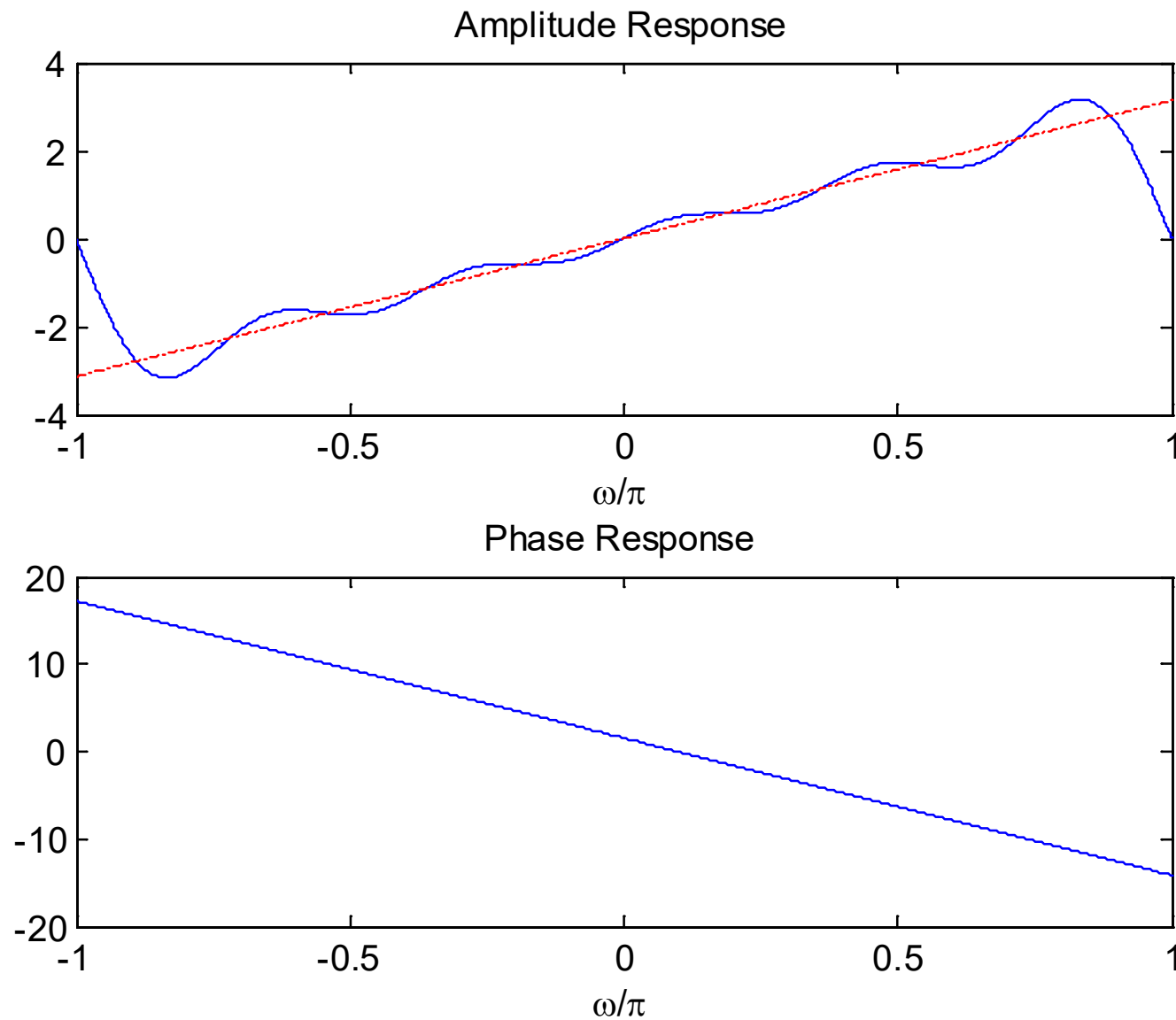


Fig.10.10: Amplitude and phase responses of differentiator

The amplitude response can be approximated as:

$$H_r(e^{j\omega}) = \left(2 \sin(\omega) - \sin(2\omega) + \frac{2}{3} \sin(3\omega) - \frac{1}{2} \sin(4\omega) + \frac{2}{5} \sin(5\omega) \right) \approx \omega$$

According to the time-shifting property, the DTFT of $h[n]$ with a time advance of 5 samples is then:

$$\begin{aligned} & \left(2 \sin(\omega) - \sin(2\omega) + \frac{2}{3} \sin(3\omega) - \frac{1}{2} \sin(4\omega) + \frac{2}{5} \sin(5\omega) \right) e^{j\pi/2} \\ & \approx \omega e^{j\pi/2} = \omega \left[\cos\left(\frac{\pi}{2}\right) + j \sin\left(\frac{\pi}{2}\right) \right] = j\omega \end{aligned}$$

This means that the causal FIR filter has a frequency response approximately equals $j\omega$ with a delay of 5 samples.

The system with frequency response $j\omega$ is known as the differentiator. Recall:

$$x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

Differentiating both sides with respect to n yields:

$$\begin{aligned} \frac{dx[n]}{dn} &= \frac{d}{dn} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) \frac{de^{j\omega n}}{dn} d\omega = \frac{1}{2\pi} \int_{-\pi}^{\pi} [X(e^{j\omega}) \cdot j\omega] e^{j\omega n} d\omega \end{aligned}$$

which confirms:

$$\frac{dx[n]}{dn} \leftrightarrow X(e^{j\omega}) \cdot j\omega$$

As a result, we expect the output $y[n]$ is close to:

$$y[n] = \frac{dx[n]}{dn} = 0.6 \cos(0.6n)$$

with a delay of 5 samples

The MATLAB program is provided as `ex10_5.m`.

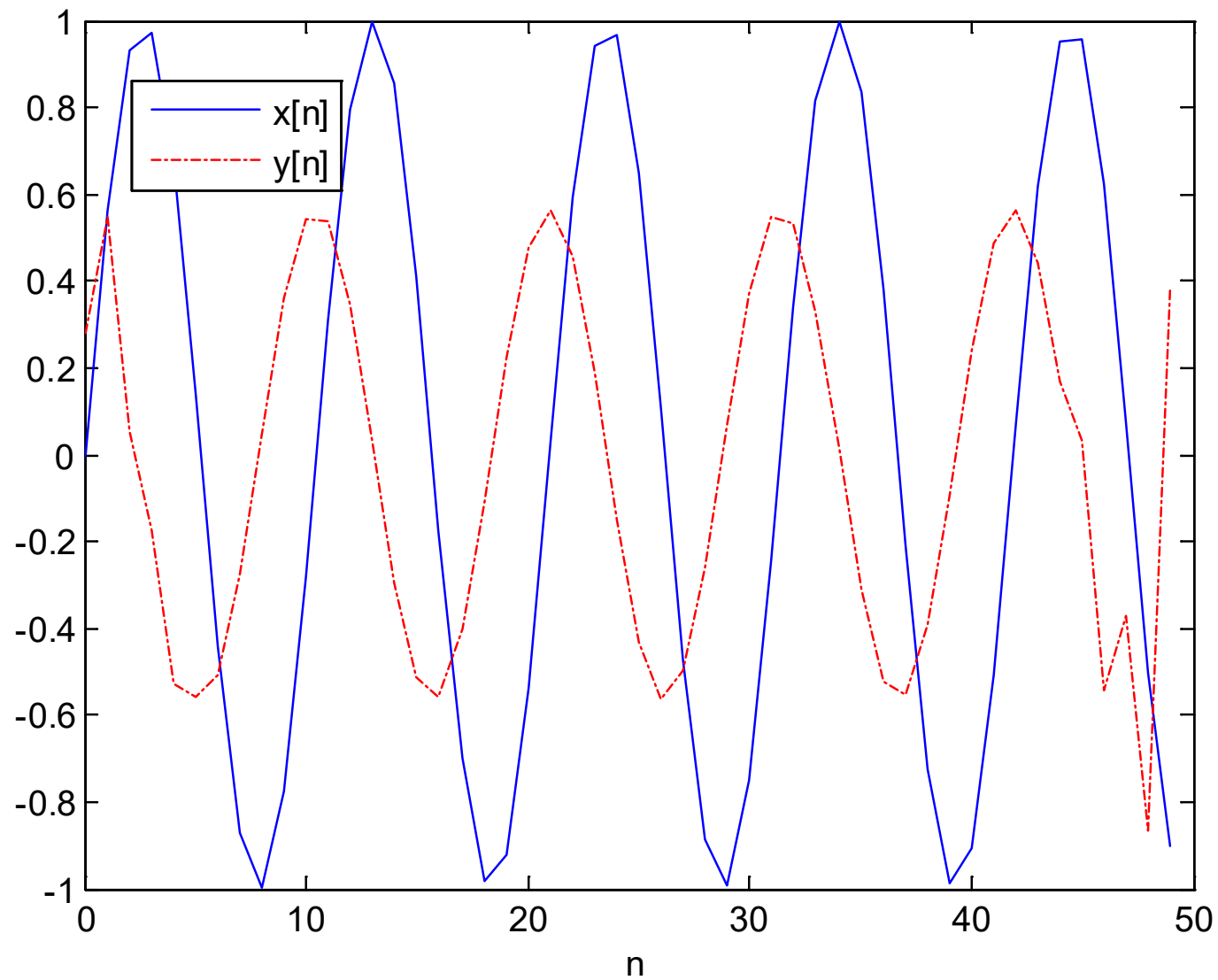


Fig.10.11: Differentiator output with sinusoidal input

Window Method

Based on **directly approximating** the desired frequency response $H_d(e^{j\omega})$

Ideal impulse response $h_d[n]$ is calculated from inverse DTFT:

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega, \quad n = \dots, -1, 0, 1, 2, \dots \quad (10.13)$$

What are the problems in using (10.13)?

Example 10.6

Determine the impulse response $h_d[n]$ of an ideal lowpass filter with a cutoff frequency of ω_c . The DTFT of the filter is:

$$H_d(e^{j\omega}) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & -\pi < \omega < -\omega_c, \quad \omega_c < \omega < \pi \end{cases}$$

Using Example 6.3, $h_d[n]$ is:

$$h_d[n] = \frac{\sin(\omega_c n)}{\pi n} = \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c n}{\pi}\right)$$

where

$$\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

Ideal filter is noncausal and its impulse response is of infinite length

The basic idea of window method is to truncate $h_d[n]$ to obtain a **linear-phase** and **causal** FIR filter via 2 steps:

- **Windowing**

- Extract a **finite** set of $h_d[n]$ with positive and negative time indexes because the impulse response is generally symmetric or anti-symmetric around $n = 0$, say, $\{h_d[-M], h_d[-M + 1], \dots, h_d[M]\}$ corresponding to a length of $(2M + 1)$
- Filter response is **linear phase** but is only an **approximation** of $H_d(e^{j\omega})$ due to the coefficient truncation

- **Time Shifting**

- **Delay** $\{h_d[-M], h_d[-M + 1], \dots, h_d[M]\}$ by M samples to obtain $h[n]$ with $h[0] = h_d[-M]$ to achieve **causality**
- Filter output is received after a **delay** of M samples

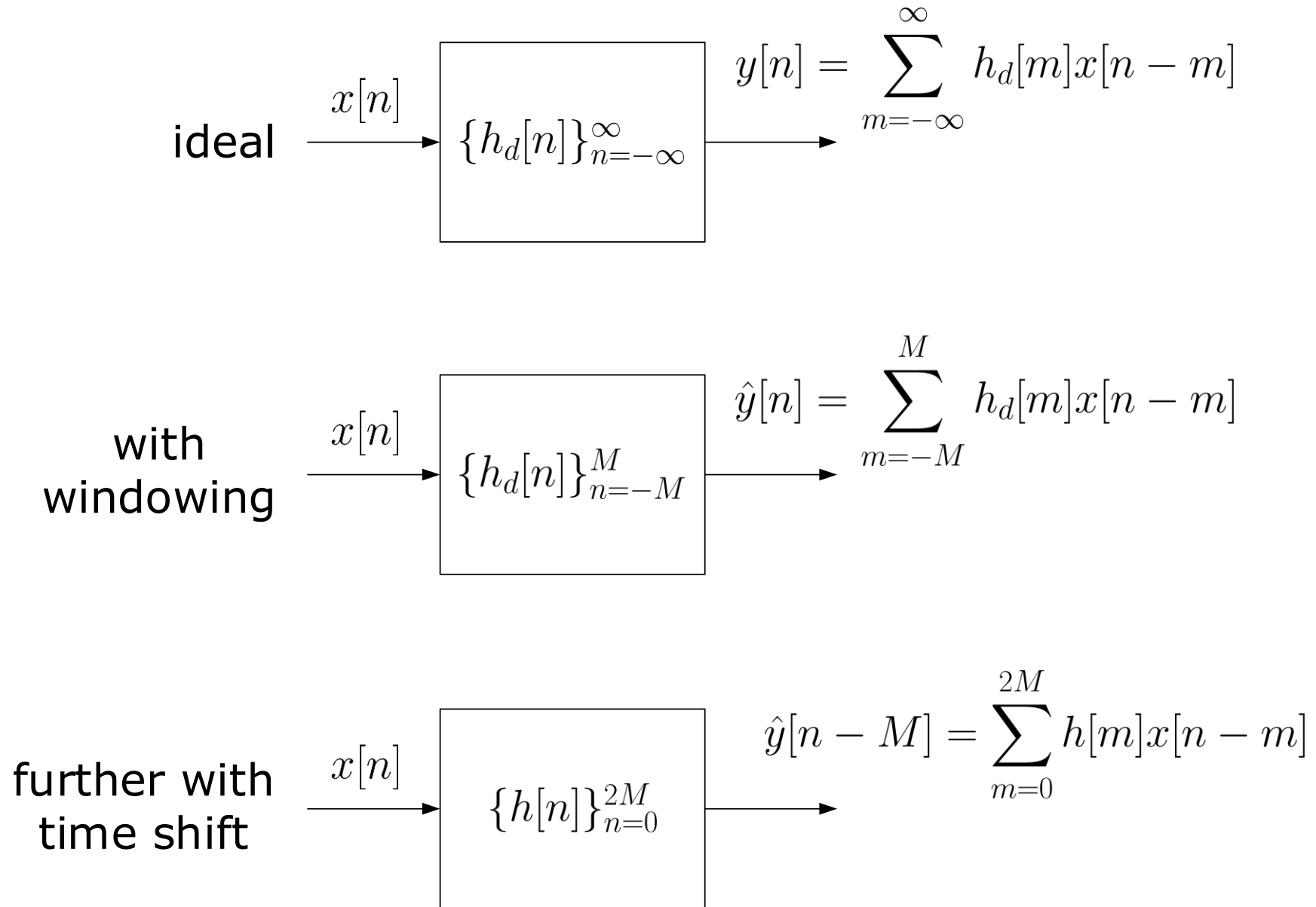


Fig.10.12: Illustration of window method

Example 10.7

Use the window method to design a linear-phase and causal FIR system with 7 coefficients to approximate an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.1\pi$.

From Example 10.6:

$$h_d[n] = \frac{\sin(0.1\pi n)}{\pi n} = 0.1\text{sinc}(0.1n)$$

Via windowing, we extract the set of $\{h_d[-3], h_d[-2], \dots, h_d[3]\}$. Notice that $h_d[n]$ is symmetric around at $n = 0$ as $h_d[n] = h_d[-n]$.

Their values are calculated as:

$$h_d[3] = h_d[-3] = \frac{\sin(0.1\pi \cdot 3)}{\pi \cdot 3} = 0.08584$$

$$h_d[2] = h_d[-2] = \frac{\sin(0.1\pi \cdot 2)}{\pi \cdot 2} = 0.09355$$

$$h_d[1] = h_d[-1] = \frac{\sin(0.1\pi)}{\pi} = 0.09836$$

$$h_d[0] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega \cdot 0} d\omega = \frac{1}{2\pi} \int_{-0.1\pi}^{0.1\pi} d\omega = 0.1$$

Via time-shifting of 3 samples, a causal filter is obtained as:

$$h[n] = h_d[n - 3], \quad n = 0, 1, \dots, 6:$$

or

$$h[n] = \{0.08584, 0.09355, 0.9836, 0.1, 0.9836, 0.09355, 0.08584\}$$

with $h[0] = 0.08584$.

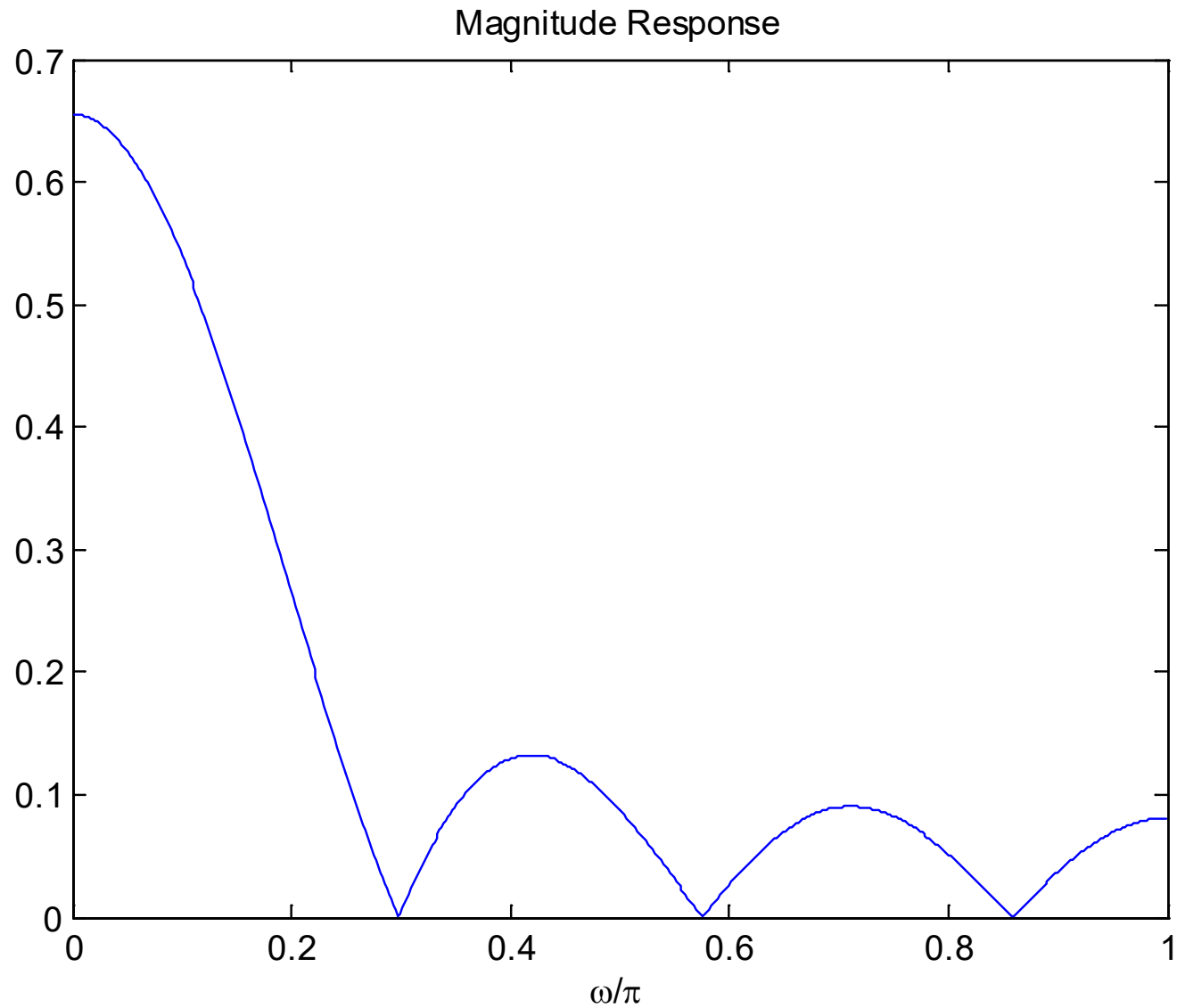


Fig.10.13: Magnitude response of lowpass filter with $N = 7$

Note that we can use the MATLAB command `fir1(6,0.1,boxcar(7),'noscale')` to produce $h[n]$

Alternatively, we can first perform the **time shifting** prior to **windowing**

As there should be a phase of $-\tau\omega$ where $\tau = (N - 1)/2$ in the practical filter, we modify the desired frequency response as:

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\omega\tau}, & |\omega| < \omega_c \\ 0, & -\pi < \omega < -\omega_c, \quad \omega_c < \omega < \pi \end{cases}$$

Note that multiplying $e^{-j\omega\tau}$ in the frequency domain corresponds to a time shift of τ in the time domain.

The corresponding impulse response is

$$h_d[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\omega}) e^{j\omega n} d\omega = \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{-j\omega\tau} e^{j\omega n} d\omega = \frac{\sin(\omega_c(n - \tau))}{\pi(n - \tau)}$$

As the filter length is N , we then perform windowing on $h_d[n]$

$$h[n] = \begin{cases} h_d[n], & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

Substituting $N = 7$ and $\omega_c = 0.1\pi$ yields the same FIR impulse response.

Note that we can base on this approach to determine $h[n]$ even when N is an even integer

The truncation operation can be considered as multiplying $h_d[n]$ by a rectangular window function:

$$w[n] = \begin{cases} 1, & 0 \leq n \leq N - 1 \\ 0, & \text{otherwise} \end{cases}$$

That is,

$$h[n] = h_d[n] \cdot w[n]$$

Generally speaking, $w[n]$ is not restricted to be rectangular and it can be any symmetric function so that the resultant filter is linear phase

Example 10.8

Use the window method to design a linear-phase and causal FIR filter of length 101 such that the sampled version of a continuous-time sinusoid with frequency of 80 Hz can pass through it with negligible attenuation while the sampled signal corresponds to a tone of frequency 120 Hz will be suppressed. The sampling frequency is 1000 Hz.

Let the continuous-time sinusoids be

$$s_i(t) = \cos(\Omega_i t), \quad \Omega_1 = 80 \cdot 2\pi, \quad \Omega_2 = 120 \cdot 2\pi$$

From (4.1), the corresponding discrete-time versions with sampling interval of $T = 1/1000$ s are

$$s_i[n] = s_i(nT) = \cos(\Omega_i n / 1000)$$

which gives:

$$s_1[n] = \cos(0.16\pi n)$$

and

$$s_2[n] = \cos(0.24\pi n)$$

The frequencies are 0.16π and 0.24π in discrete-time domain.

In order to suppress $s_2[n]$ while keeping $s_1[n]$, we can use a lowpass filter with cutoff frequency of $\omega_c = 0.2\pi$, which is simply the average of the two discrete frequencies.

Using Example 10.7 with $N = 101$ and $\omega_c = 0.2\pi$, the required filter impulse response $h[n]$ is:

$$h[n] = \frac{\sin(0.2\pi(n - 50))}{\pi(n - 50)} = 0.2\text{sinc}(0.2(n - 50)), \quad 0 \leq n \leq 100$$

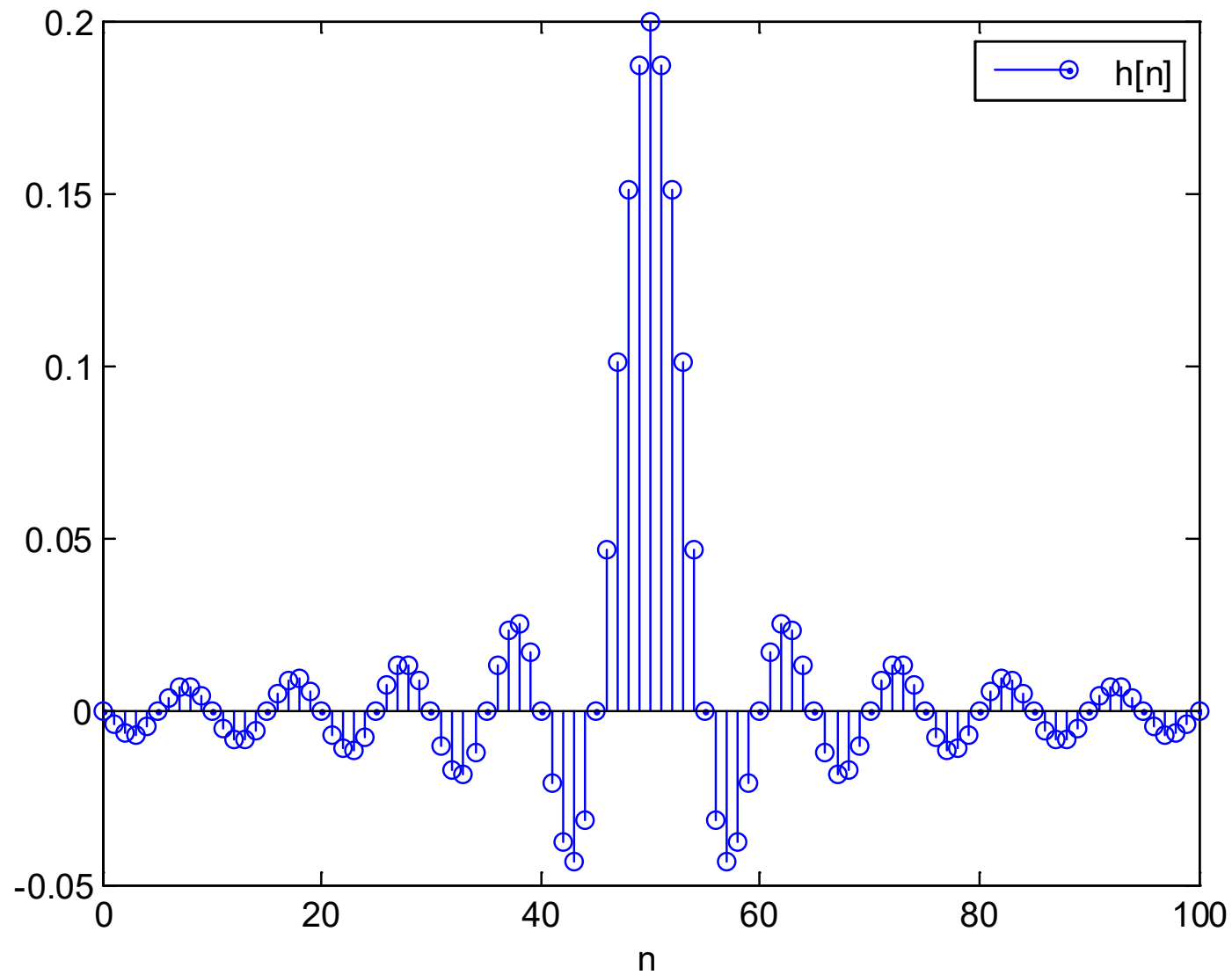


Fig.10.14: Impulse response of lowpass filter with $N = 101$

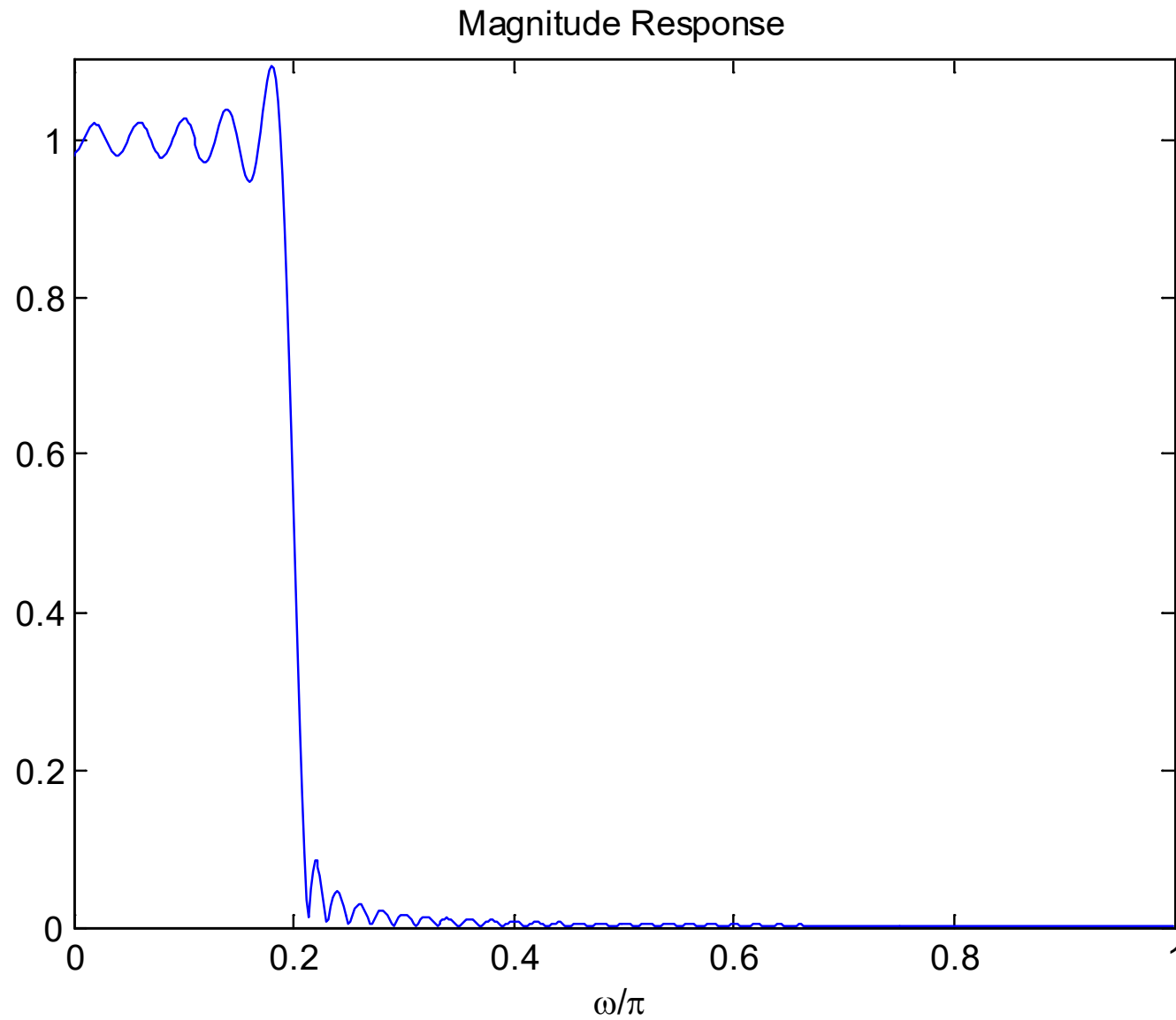


Fig.10.15: Magnitude response of lowpass filter with $N = 101$

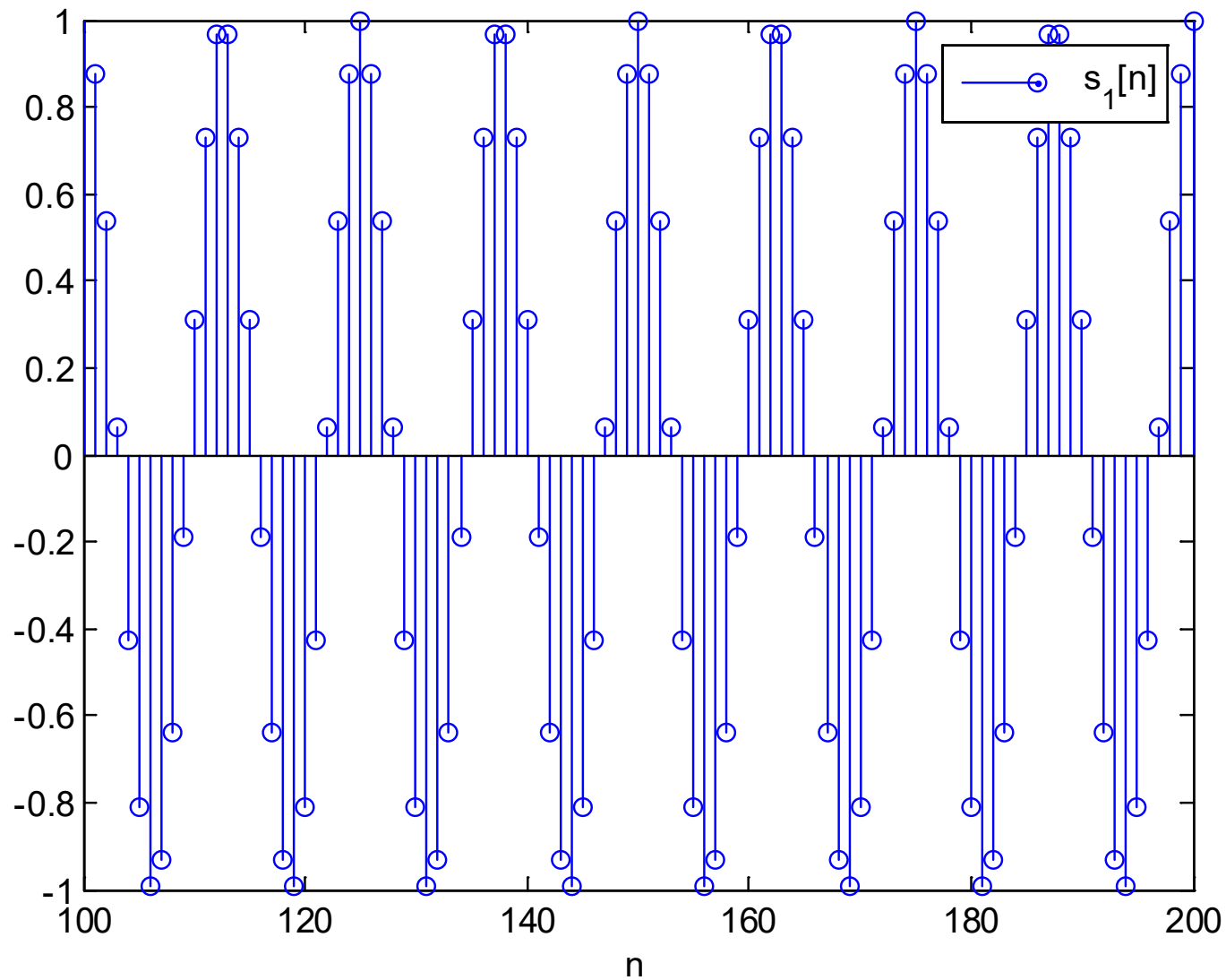


Fig.10.16: Discrete-time tone with frequency 0.16π



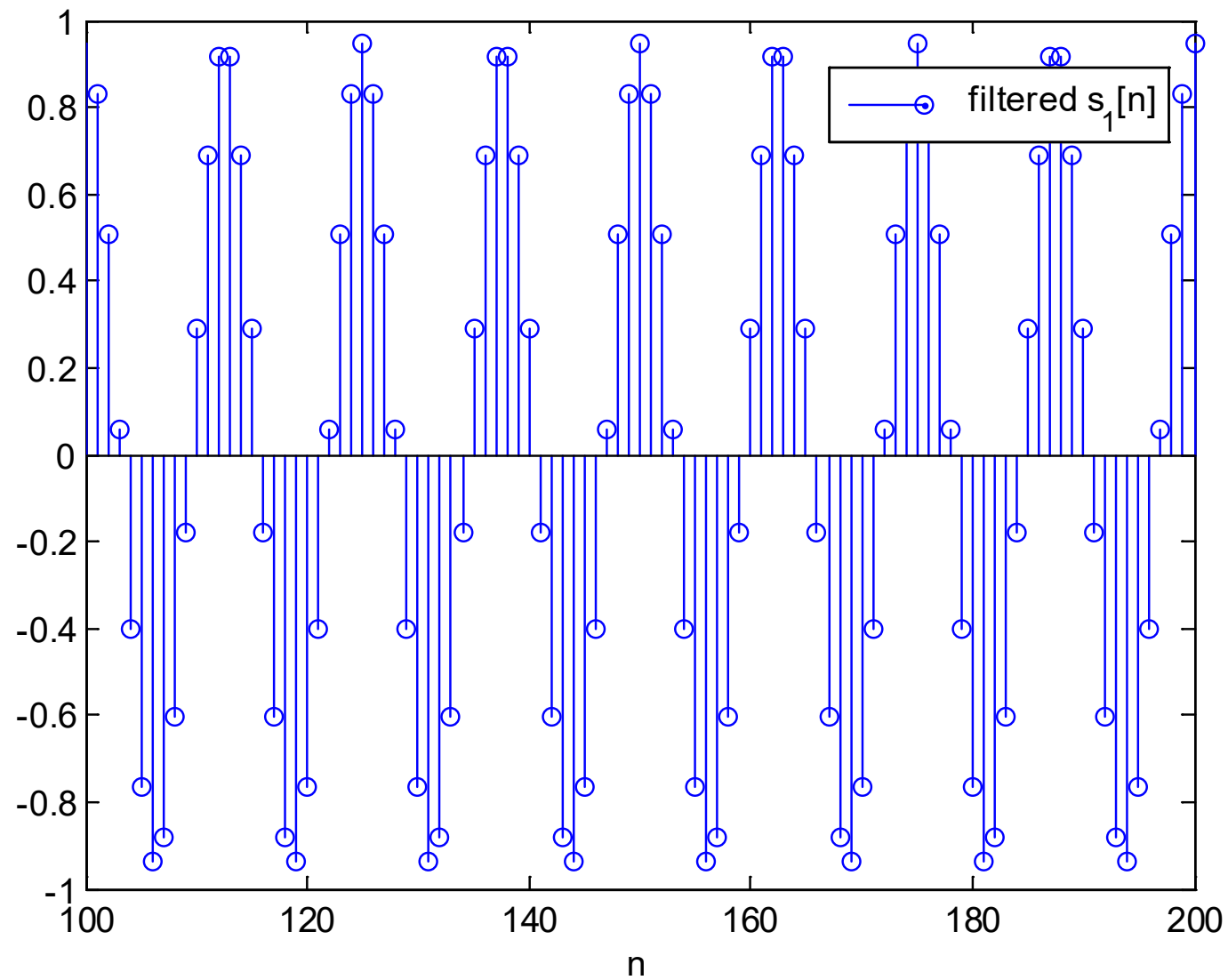


Fig.10.18: Filter output for frequency 0.16π

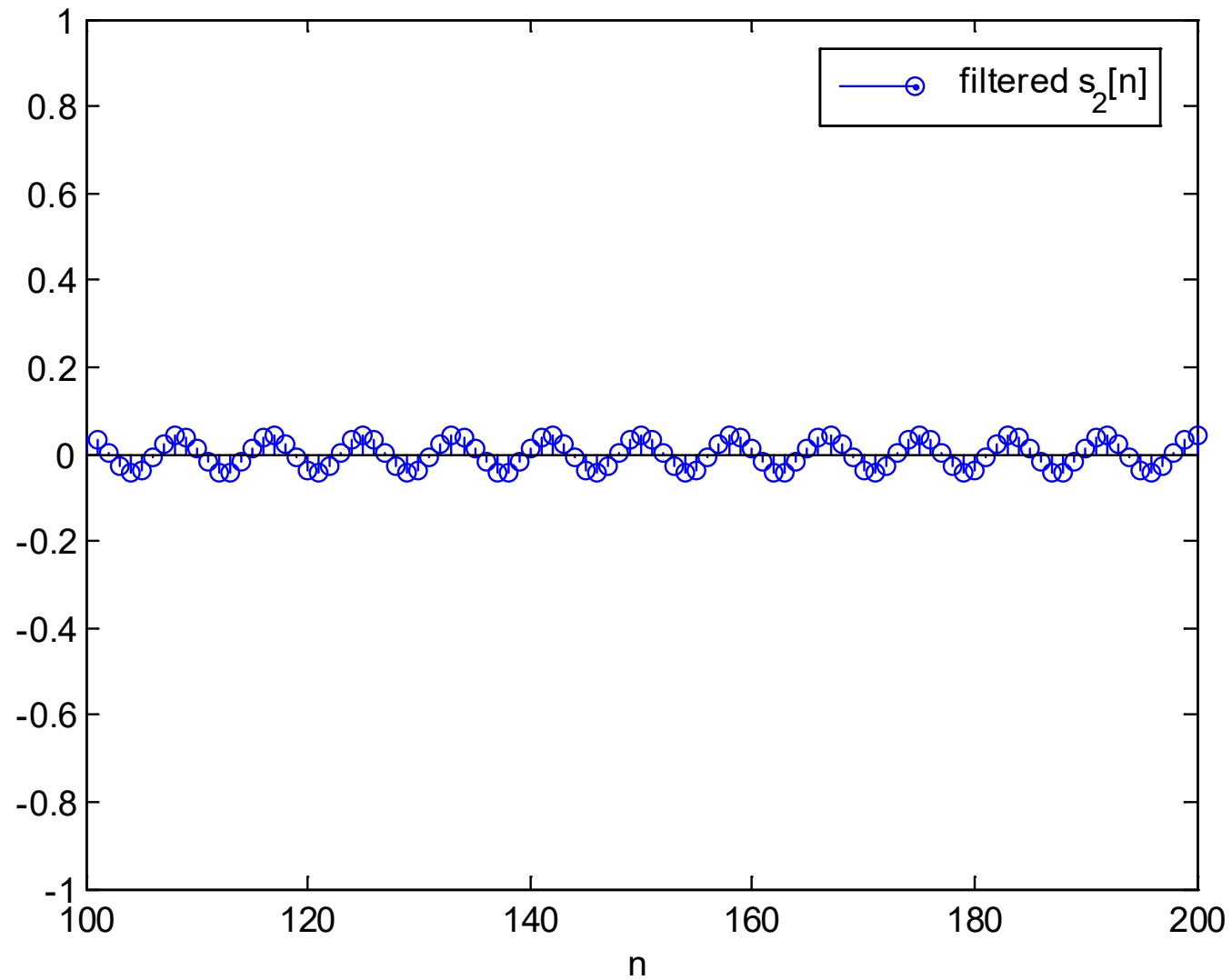


Fig.10.19: Filter output for frequency 0.24π

Analysis of Windowing

To incorporate **passband** and **stopband frequencies** and **ripples** in the FIR filter design, we need to study the windowing effect

Recall

$$h[n] = h_d[n] \cdot w[n] \quad (10.14)$$

According to the multiplication property of (6.18):

$$H(e^{j\omega}) = H_d(e^{j\omega}) \tilde{\otimes} W(e^{j\omega}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H_d(e^{j\tau}) W(e^{j(\omega-\tau)}) d\tau \quad (10.15)$$

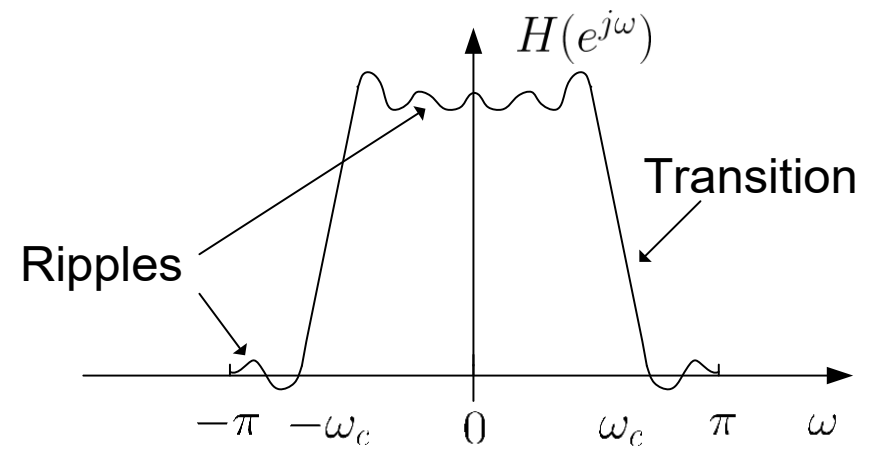
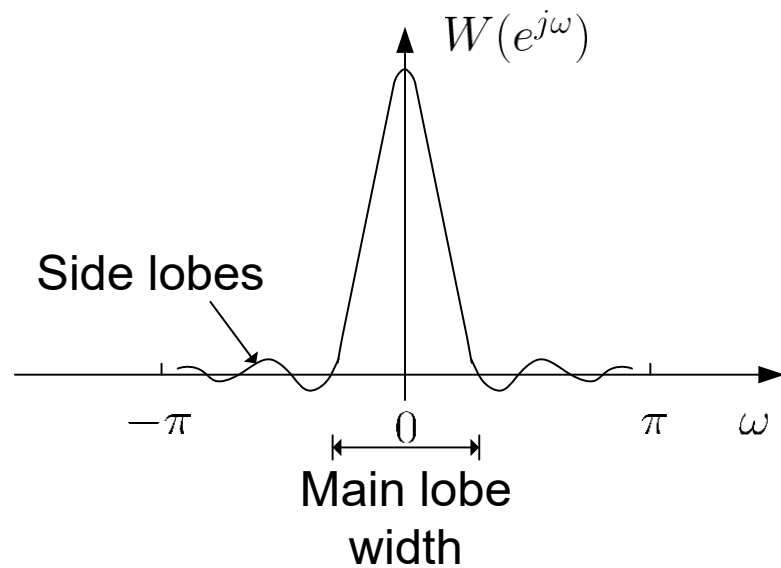
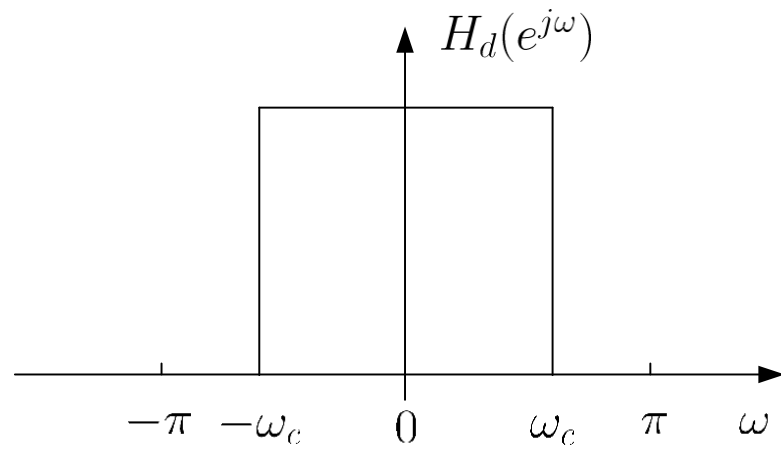


Fig.10.20: Illustration of $H(e^{j\omega}) = H_d(e^{j\omega}) \tilde{\otimes} W(e^{j\omega})$

$H(e^{j\omega})$ is a **smeared** version of ideal $H_d(e^{j\omega})$

$W(e^{j\omega})$ has a **peaky main lobe** and several **side lobes** of smaller magnitudes

Main lobe produces the **transition band** in $H(e^{j\omega})$. That is, the transition width is proportional to the main lobe width, which is inversely proportional to the filter length N

Side lobes are responsible to produce the **ripples** in the passband and stopband with $\delta_1 = \delta_2$

$H(e^{j\omega})$ approaches $H_d(e^{j\omega})$ when $W(e^{j\omega})$ has the smallest main lobe width and side lobe magnitude. That is, the **ideal** but not practical form of $W(e^{j\omega}) = \delta(\omega)$ gives $H(e^{j\omega}) = H_d(e^{j\omega})$

- Rectangular Window

$$w[n] = u[n] - u[n - N] \quad (10.16)$$

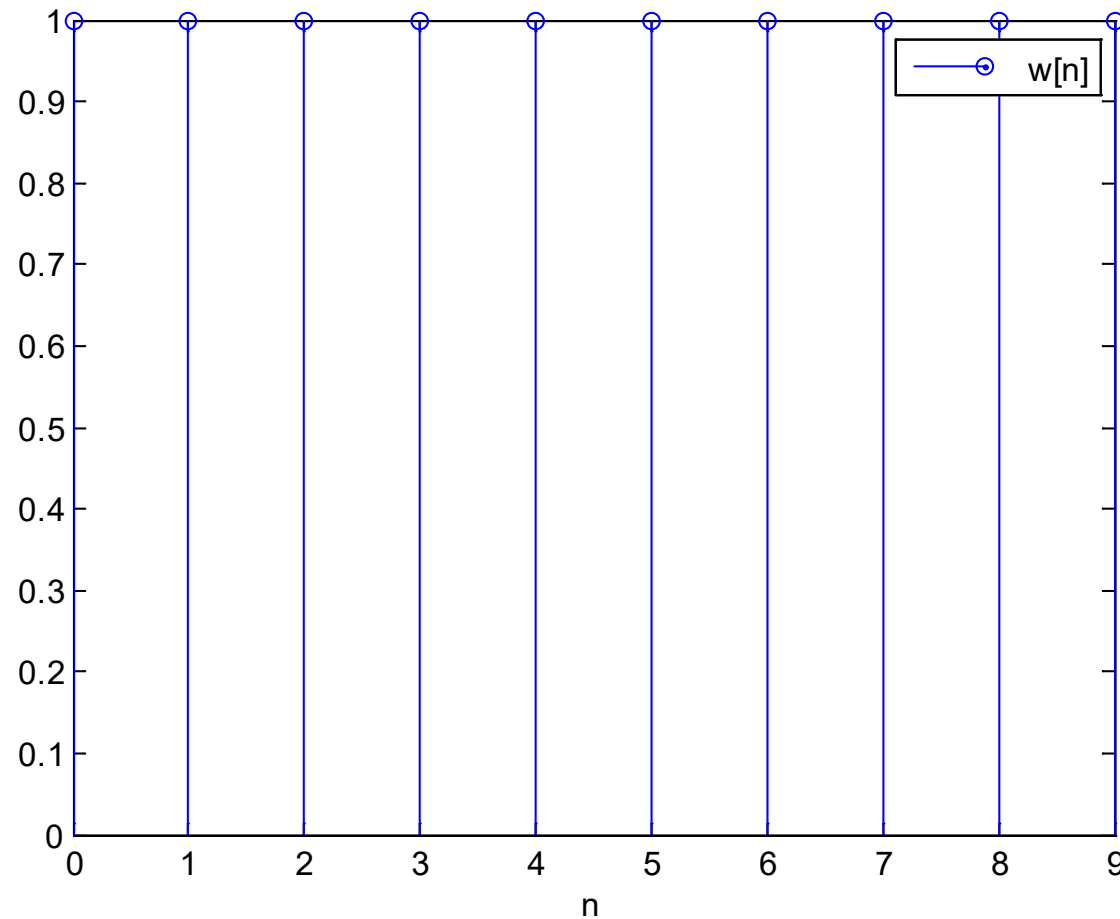


Fig.10.21: Rectangular window function for $N = 10$

From Example 10.1:

$$W(e^{j\omega}) = \frac{\sin(\omega N/2)}{\sin(\omega/2)} \cdot e^{-j\omega(N-1)/2} \quad (10.17)$$

The **main lobe width** is

$$\Delta\omega_m = \frac{4\pi}{N} \quad (10.18)$$

which is proportional to the filter transition width

That is, transition width decreases as the filter length N increases.

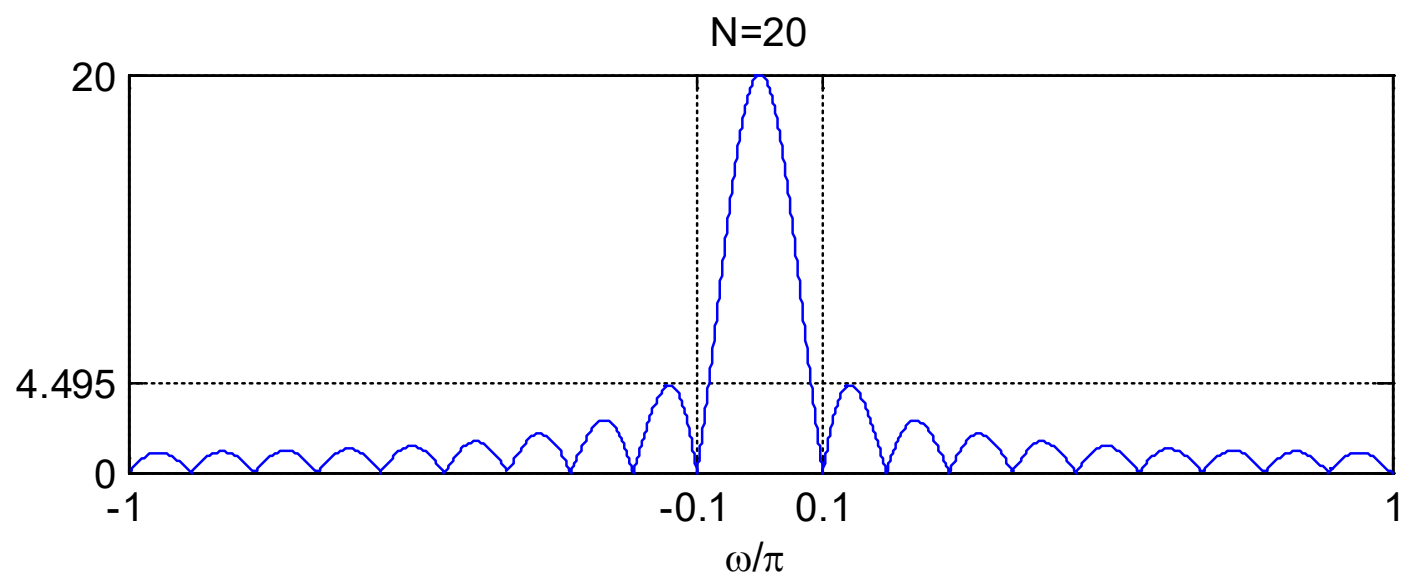
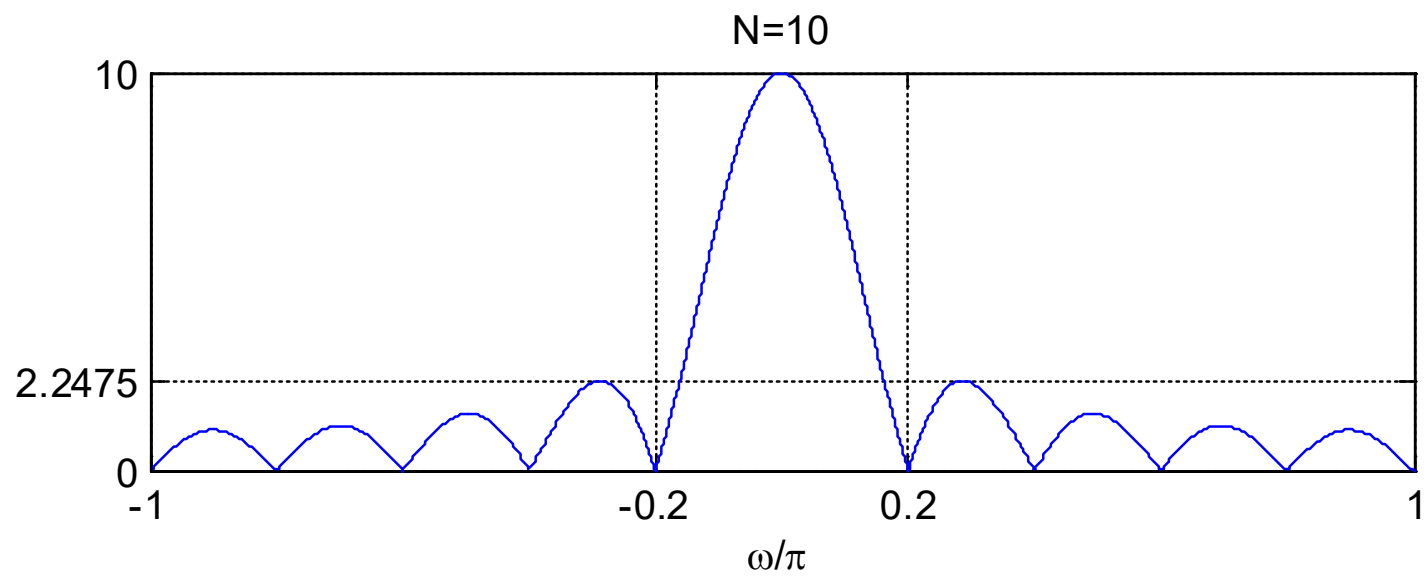


Fig.10.22: Magnitude responses of rectangular window for $N = 10$ & $N = 20$
Relative peak side lobe, r_p , is the ratio of the peak side lobe height to main lobe height in dB:

$$r_p = 20 \log_{10} \left(\frac{2.2475}{10} \right) = 20 \log_{10} \left(\frac{4.495}{20} \right) = -13 \text{ dB} \quad (10.19)$$

A larger r_p implies ripples with bigger magnitudes

As N increases, widths of the main lobe and all side lobes decrease but their areas remain unchanged, meaning that the ripples cannot be reduced.

This is known as **Gibbs phenomenon** due to sudden change in transition from 0 to 1 and 1 to 0. This can be removed by tapering the window smoothly to zero at each end. However, the heights of the side lobes can be diminished but at the

expense of wider main lobe and hence a wider transition.

- **Bartlett** Window

$$w[n] = \begin{cases} 1 - \frac{2 \left| n - \frac{N-1}{2} \right|}{N-1}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (10.20)$$

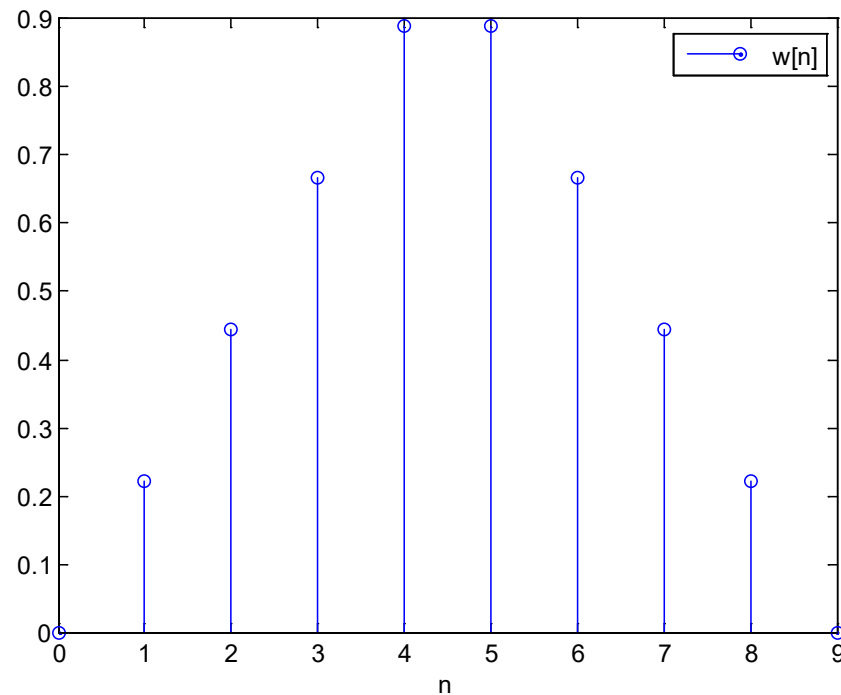


Fig.10.23: Bartlett window function for $N = 10$

- **Hanning** Window

$$w[n] = \begin{cases} \frac{1}{2} \left(1 - \cos \left(\frac{2\pi n}{N-1} \right) \right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (10.21)$$

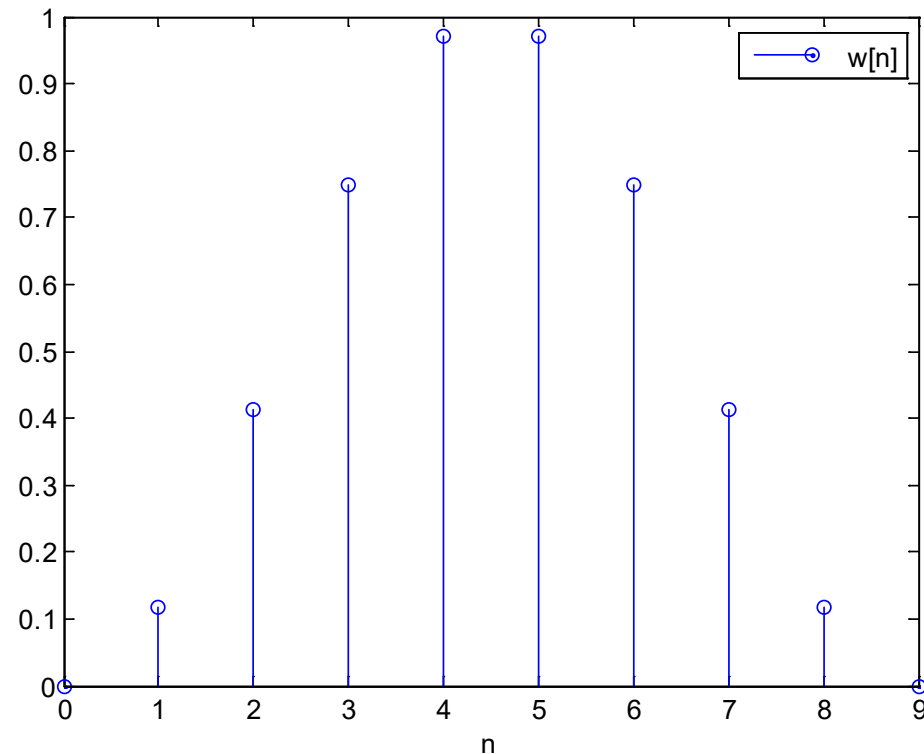


Fig.10.24: Hanning window function for $N = 10$

- **Hamming** Window

$$w[n] = \begin{cases} 0.54 - 0.46 \cos\left(\frac{2\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (10.22)$$

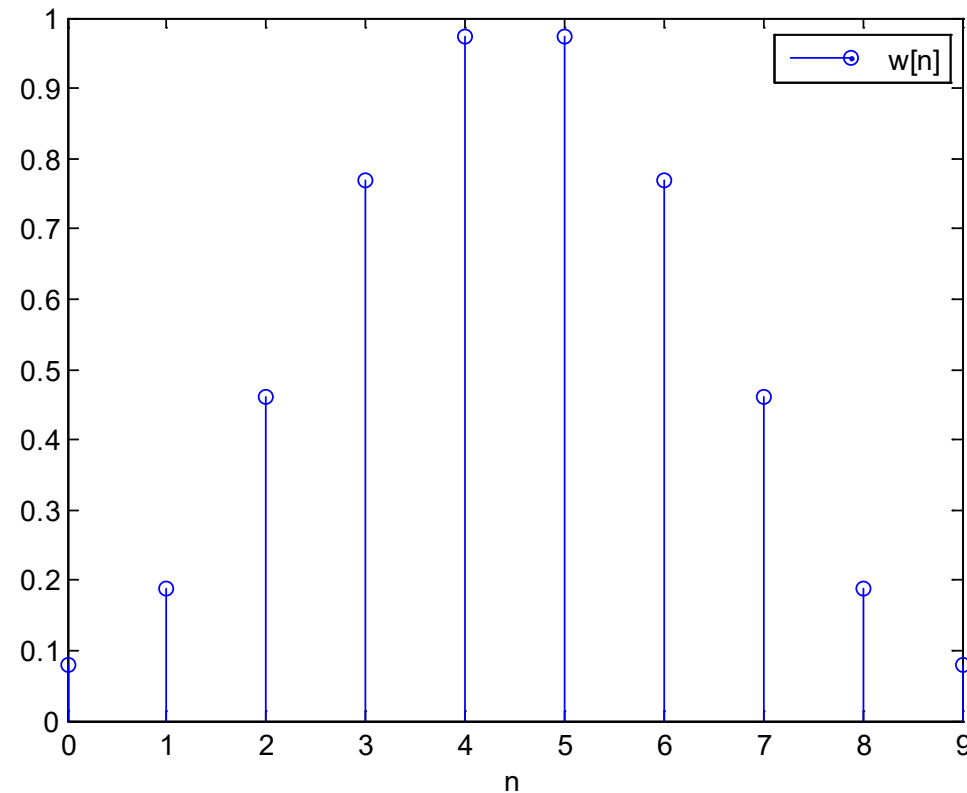


Fig.10.25: Hamming window function for $N = 10$

- **Blackman** Window

$$w[n] = \begin{cases} 0.42 - 0.5 \cos\left(\frac{2\pi n}{N-1}\right) + 0.08 \cos\left(\frac{4\pi n}{N-1}\right), & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (10.23)$$

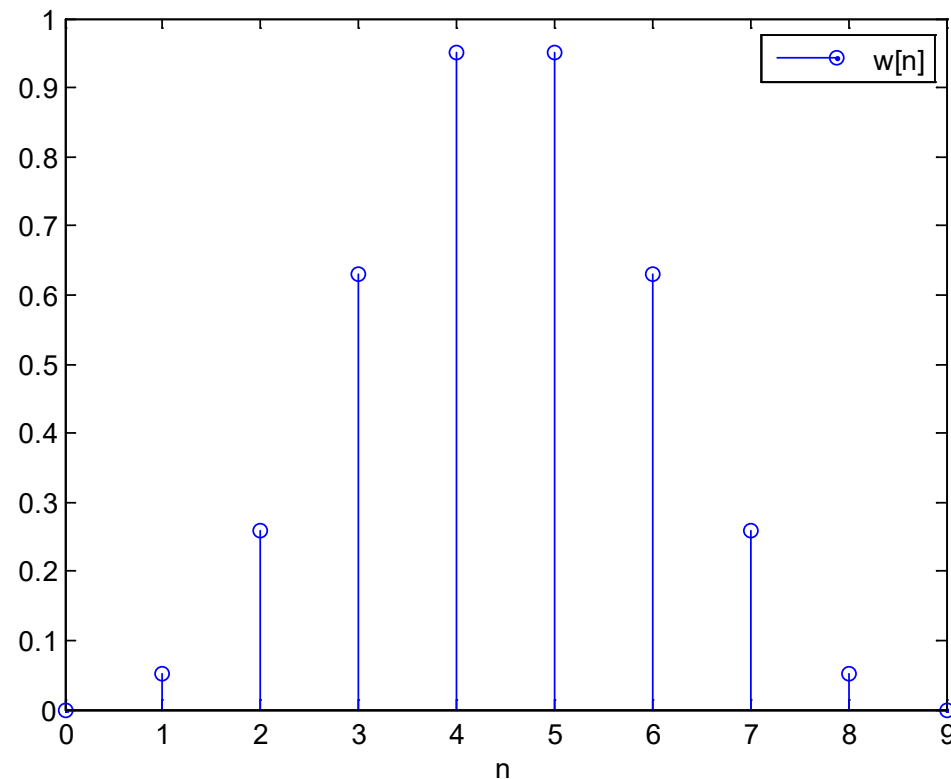


Fig.10.26: Blackman window function for $N = 10$

Note that by using $(N + 2)$ terms for a filter length of N and then discarding the end points, we can avoid zero coefficients at both ends for Bartlett, Hanning and Blackman windows.

Window	$\Delta\omega_m$	r_p (dB)
Rectangular	$4\pi/N$	-13
Bartlett	$8\pi/N$	-27
Hanning	$8\pi/N$	-32
Hamming	$8\pi/N$	-43
Blackman	$12\pi/N$	-58

Table 10.1: Characteristics of window functions

As a result, there is a tradeoff between transition and ripple

Example 10.9

Use the window method to design a linear-phase and causal FIR system to approximate an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.5\pi$. Try the rectangular and Bartlett window functions with filter lengths of 7, 21, 61 and 101, to investigate the frequency magnitude responses.

Using Example 10.7, $h[n]$ with the rectangular window is:

$$h[n] = \frac{\sin(\omega_c(n - \tau))}{\pi(n - \tau)}, \quad 0 \leq n \leq N - 1$$

with N equals 7, 21, 61 and 101 and $\tau = (N - 1)/2$.

Multiplying $h[n]$ by (10.20) yields the corresponding filter coefficients for the Bartlett window.

The MATLAB program is provided as `ex10_9.m`.

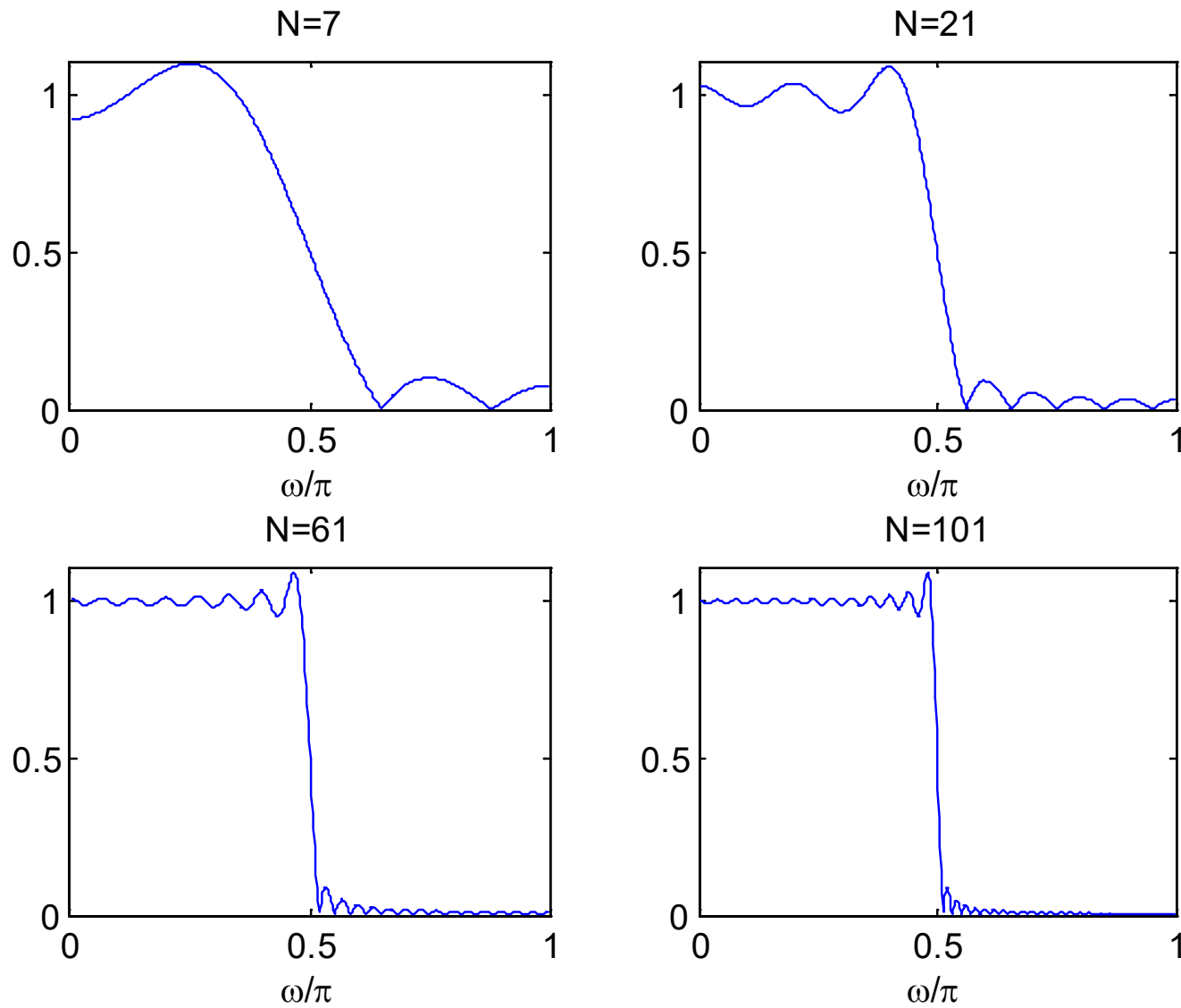


Fig.10.27: Magnitude responses with rectangular window at different N

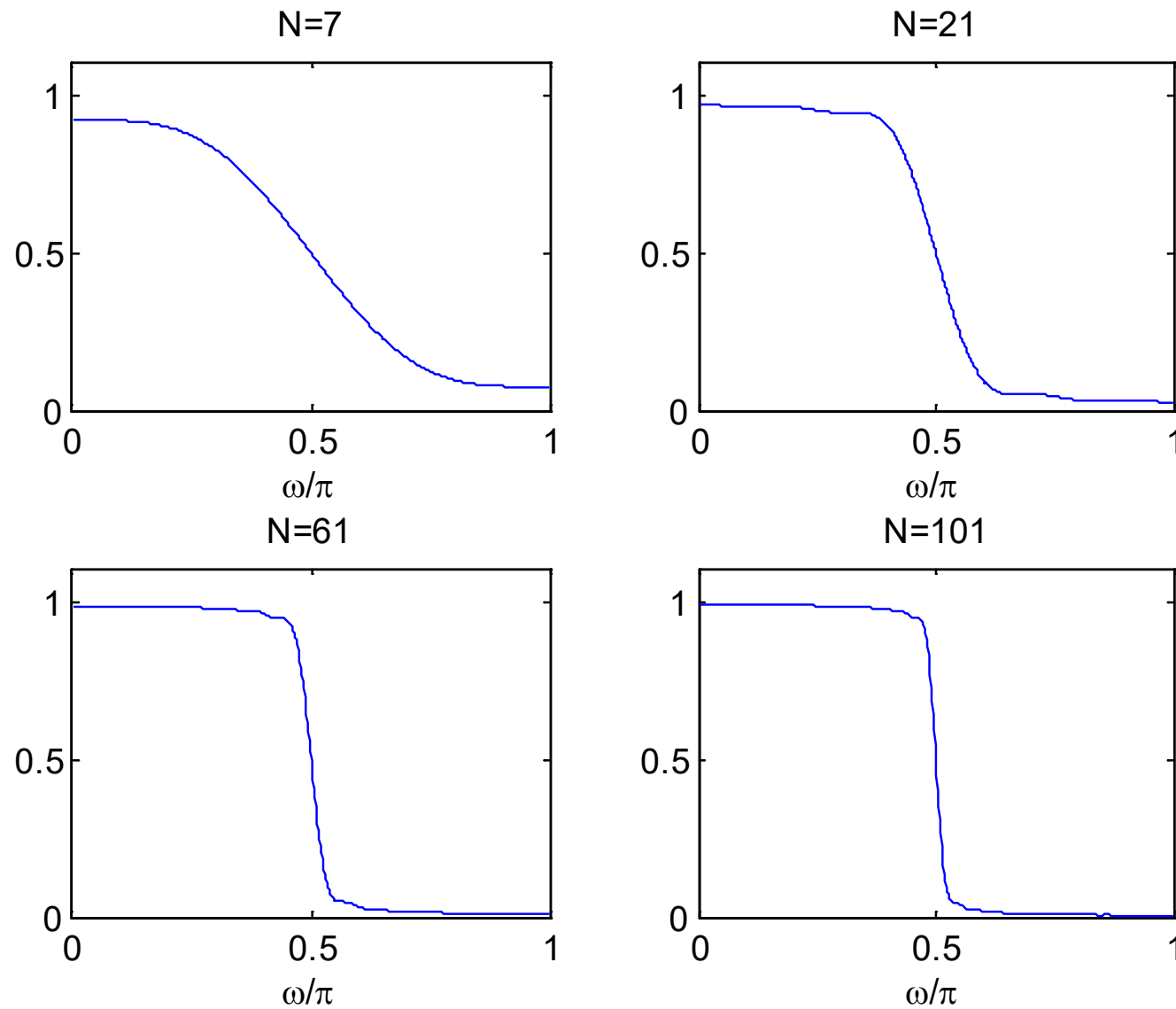


Fig.10.28: Magnitude responses with Bartlett window at different N

Example 10.10

Use the window method to find the impulse response of a linear-phase and causal FIR filter which approximates an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.25\pi$. It is required that the filter is of fourth-order and Bartlett window is employed.

A fourth-order filter implies a filter length of $N = 5$. Using Example 10.7 with $N = 5$ and $\omega_c = 0.25\pi$:

$$h[n] = \frac{\sin(\omega_c(n-2))}{\pi(n-2)} \cdot w[n], \quad 0 \leq n \leq 4$$

To avoid zero coefficients, we set $N = 7$ in (10.20) and extract the middle 5 values to yield $w[n] = \{1/3, 2/3, 1, 2/3, 1/3\}$ with

$$w[0] = 1/3.$$

As a result

$$h[n] = \{0.0531, 0.1501, 0.25, 0.1501, 0.0531\}$$

with $h[0] = 0.0531$

$H(z)$ is:

$$H(z) = 0.0531 + 0.1501z^{-1} + 0.25z^{-2} + 0.1501z^{-3} + 0.0531z^{-4}$$

The MATLAB command `fir1(4, 0.25, triang(5), 'noscale')` can also produce $h[n]$.

Note that if we use $N = 5$ in (10.20), $w[n] = \{0, 1/2, 1, 1/2, 0\}$ with $w[0] = 0$. The resultant impulse response will be $h[n] = \{0, 0.1125, 0.25, 0.1125, 0\}$ with $h[0] = 0$, indicating that the

effective length is only 3

In window method, the **ripple** is:

$$\delta = \delta_1 = \delta_2$$

And the **transition width** is:

$$\Delta\omega = \omega_s - \omega_p \quad \text{with} \quad \omega_c = (\omega_p + \omega_s)/2$$

or

$$\omega_p = \omega_c - \Delta\omega/2 \quad \text{and} \quad \omega_s = \omega_c + \Delta\omega/2$$

They are determined from $H(e^{j\omega}) = H_d(e^{j\omega}) \tilde{\otimes} W(e^{j\omega})$ in (10.15), which is not straightforward for computation

Nevertheless, by noting that (10.15) corresponds to one of the forms in (10.9)-(10.12), $H_r(e^{j\omega})$ is easily obtained by

extracting the amplitude

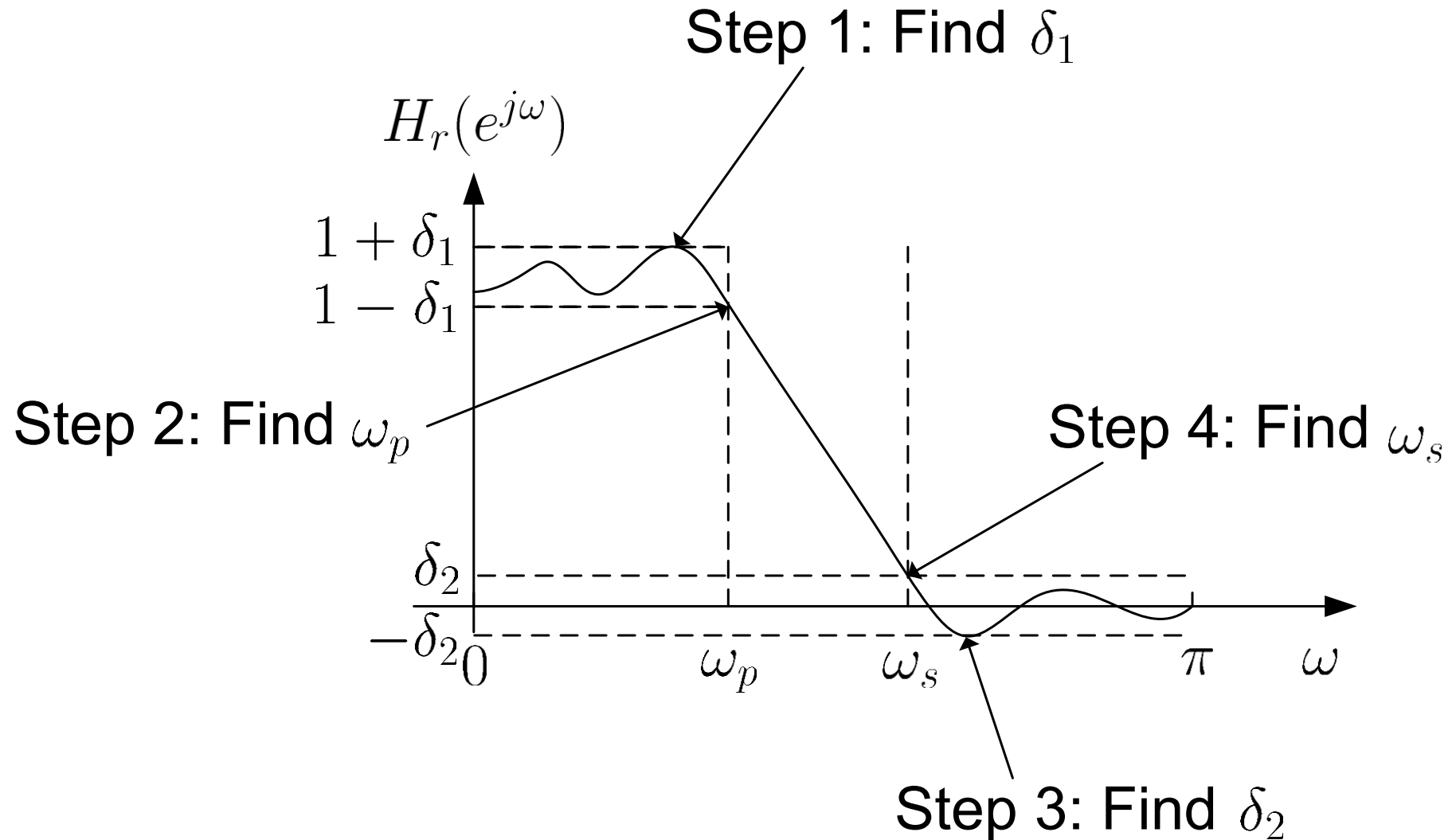


Fig.10.29: Steps to find passband and stopband ripples and frequencies

Example 10.11

Use the window method with the rectangular window to find the impulse response of a linear-phase and causal FIR filter which approximates an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.5\pi$. It is required that the filter has a length of $N = 21$. Plot its amplitude response and then determine δ_1 , δ_2 , ω_p , ω_s and $\Delta\omega$.

Using Example 10.7 with $N = 21$ and $\omega_c = 0.5\pi$, $h[n]$ is:

$$h[n] = \frac{\sin(0.5\pi(n - 10))}{\pi(n - 10)}, \quad 0 \leq n \leq 20$$

As $h[n]$ is symmetric and N is odd, from (10.9):

$$H_r(e^{j\omega}) = 2 \sum_{n=0}^9 h[n] \cos((10 - n)\omega) + h[10]$$

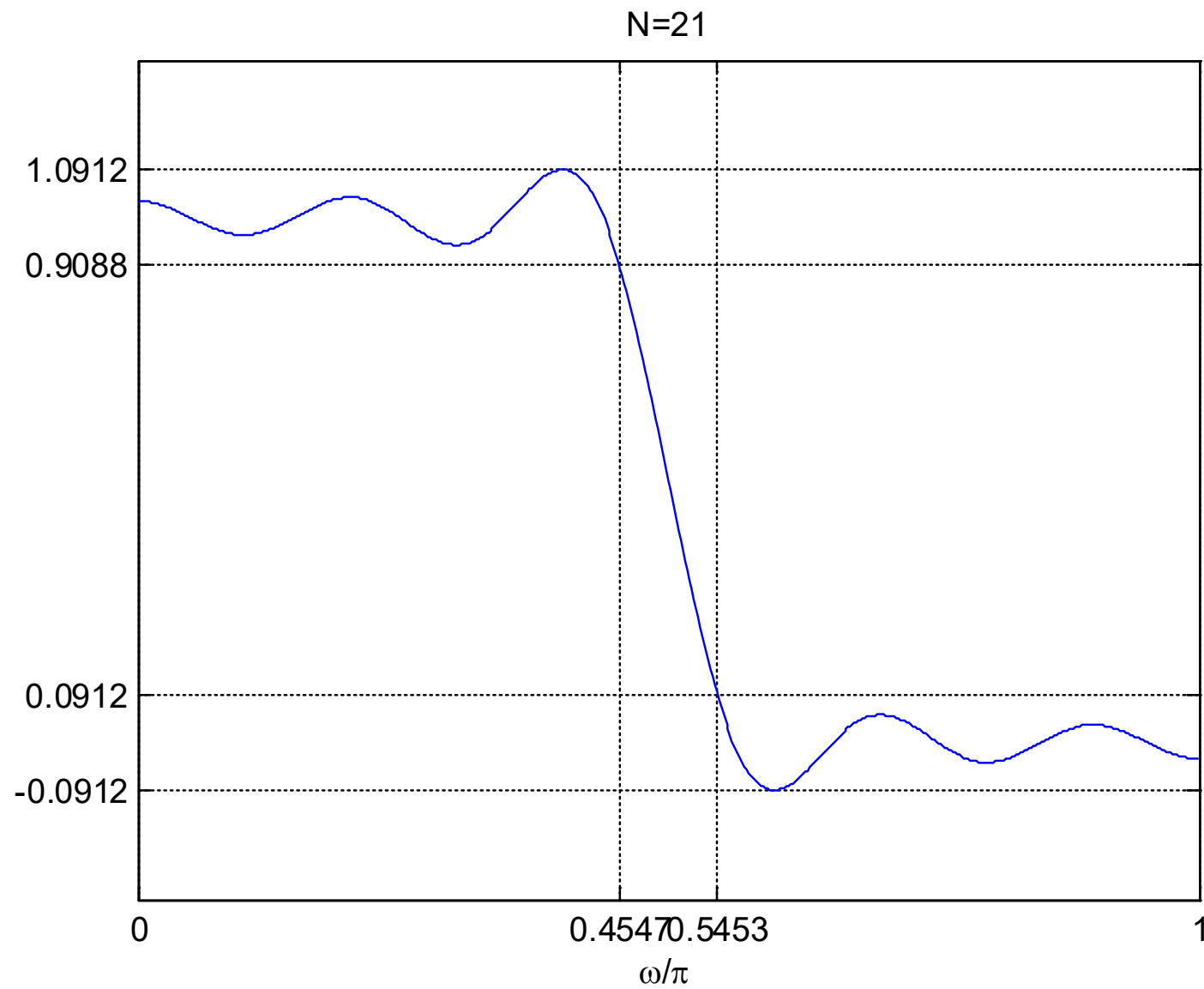


Fig.10.30: Measuring parameters from $H_r(e^{j\omega})$

We see that

$$\delta = \delta_1 = \delta_2 = 0.0912$$

Peak approximation error, e_p , which is the ripple in dB, is:

$$e_p = 20 \log_{10} \delta = 20 \log_{10} 0.0912 = -21 \text{ dB}$$

Furthermore

$$\omega_p = 0.4547\pi$$

and

$$\omega_s = 0.5453\pi$$

which gives

$$\Delta\omega = \omega_s - \omega_p = 0.2846$$

with

$$\omega_c = (\omega_p + \omega_s)/2 = 0.5\pi$$

The MATLAB program is provided as `ex10_11.m`.

Window	$\Delta\omega$	e_p (dB)
Rectangular	$1.8\pi/N$	-21
Bartlett	$6.1\pi/N$	-25
Hanning	$6.2\pi/N$	-44
Hamming	$6.6\pi/N$	-53
Blackman	$11\pi/N$	-74

Table 10.2: Transition width and ripple due to different window functions

Example 10.12

Use the window method to design a linear-phase and causal FIR filter which approximates an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.5\pi$. The maximum allowable transition width is $\Delta\omega = 0.05\pi$ and the maximum allowable tolerance is $\delta = 0.005$.

The ripple of $\delta = 0.005$ corresponds to:

$$e_p = 20 \log_{10}(0.005) = -46 \text{ dB}$$

From Table 10.2, Hamming and Blackman windows are the two candidates which can meet the ripple requirement

We choose the former because it involves a shorter filter length. The required length for the Hamming window is:

$$\Delta\omega = 0.05\pi = 6.6\pi/N \Rightarrow N = 132$$

Using Example 10.7 with $N = 132$ and $\omega_c = 0.5\pi$, the filter impulse response is:

$$h[n] = \frac{\sin(0.5\pi(n - 65.5))}{\pi(n - 65.5)} \cdot w[n], \quad 0 \leq n \leq 131$$

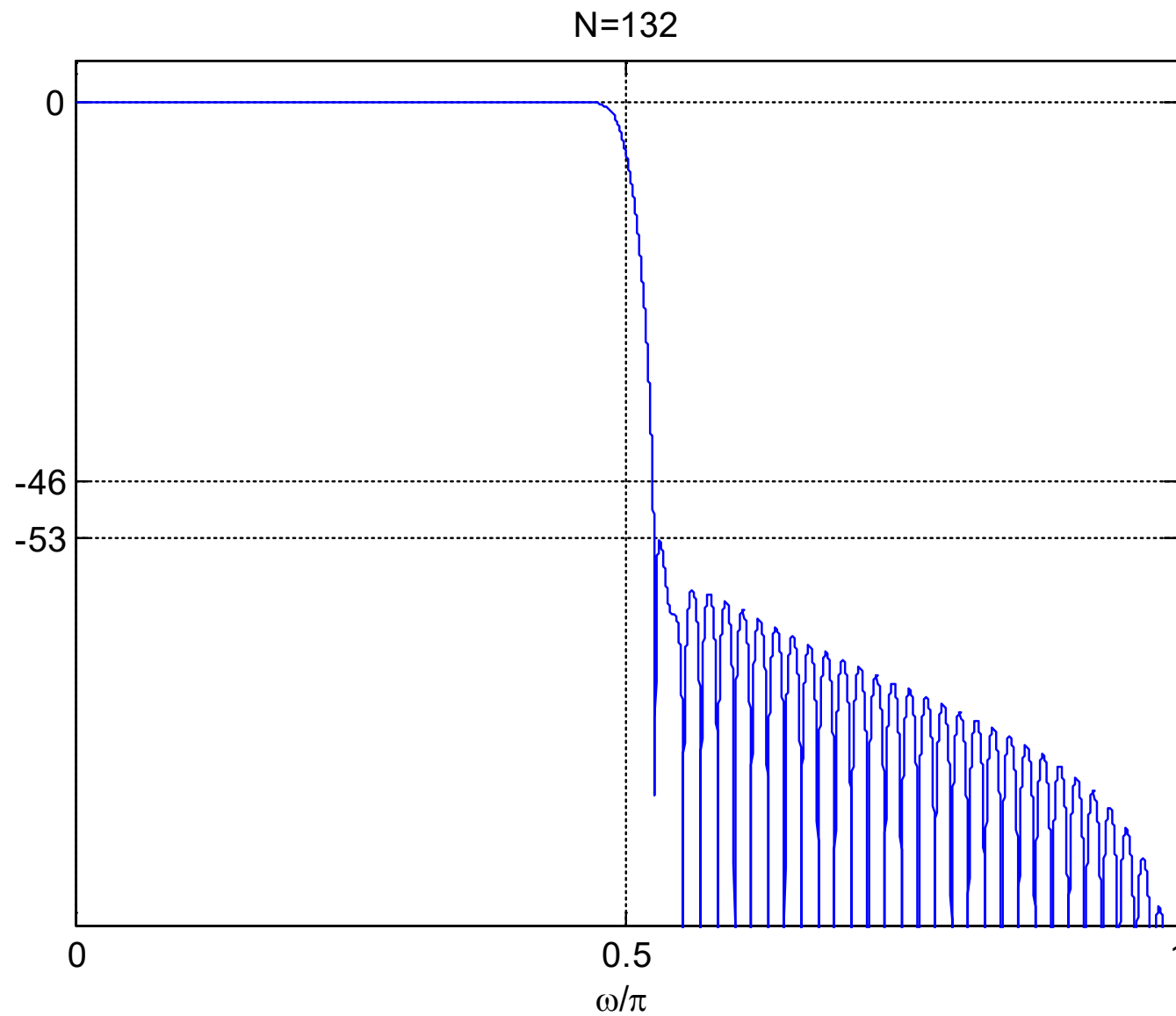


Fig.10.31: Magnitude response with Hamming window

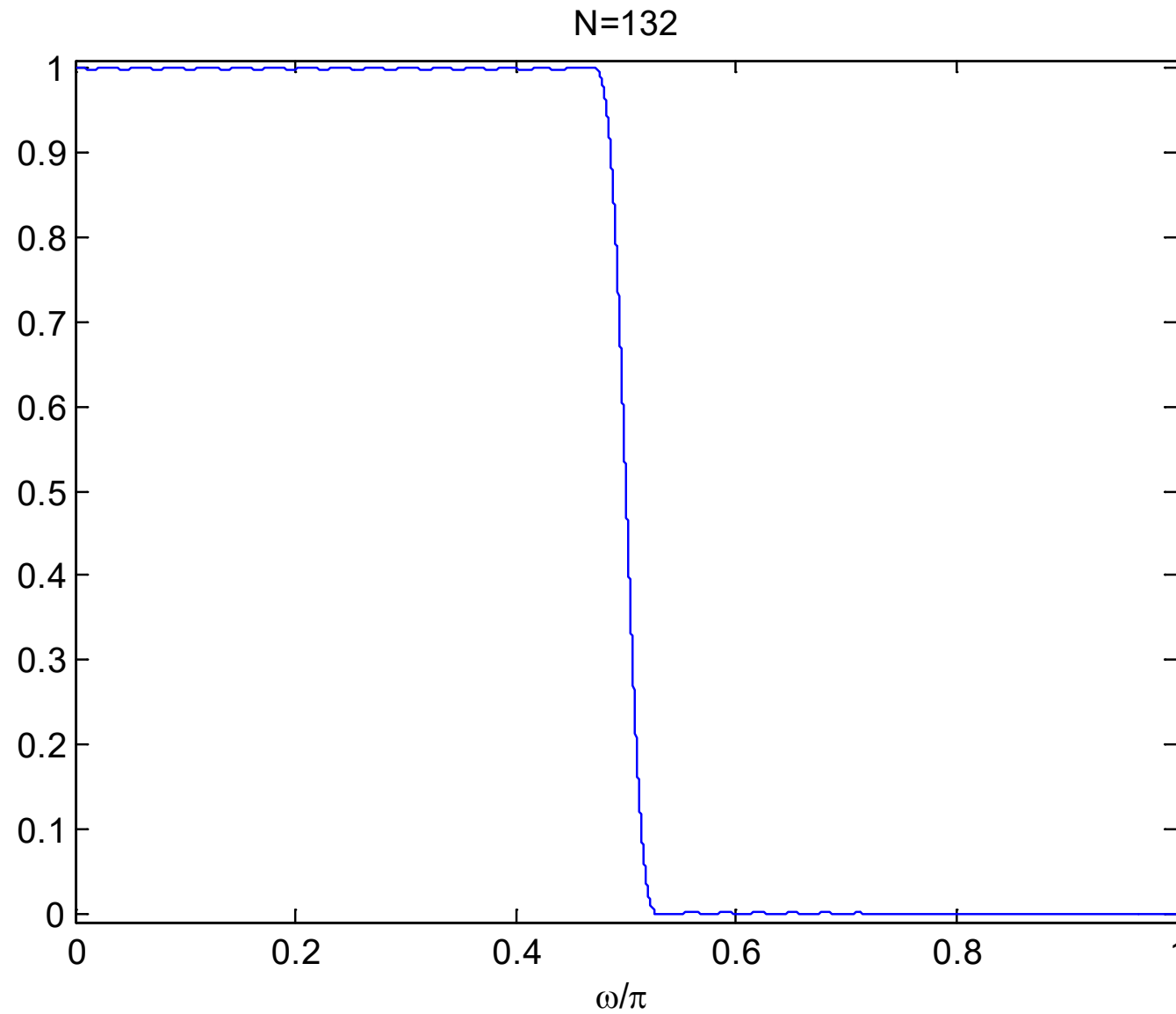


Fig.10.32: Amplitude response with Hamming window

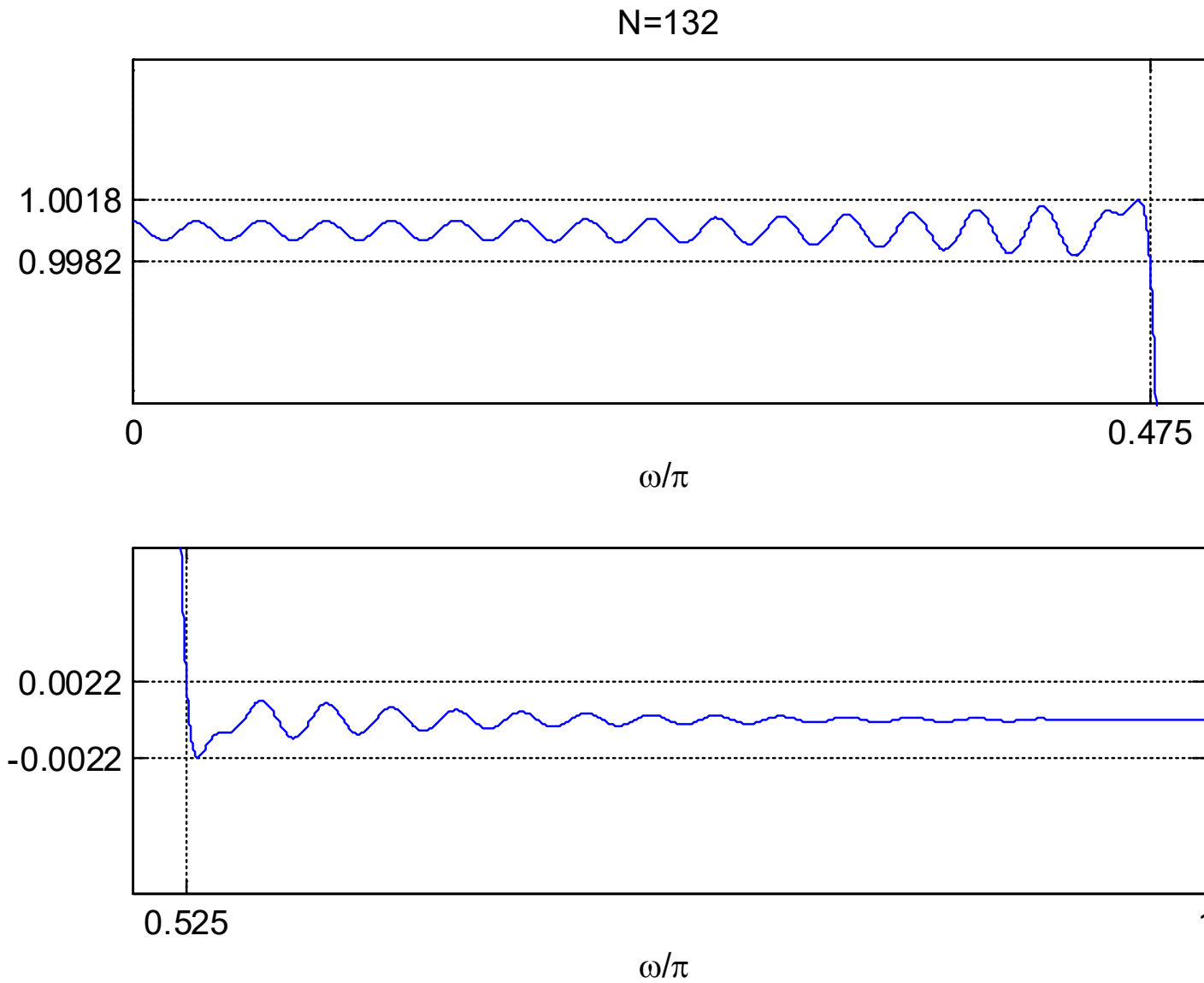


Fig.10.33: Zoomed amplitude responses with Hamming window

In summary:

$$e_p = -53 \text{ dB} \text{ or } \delta \approx 0.002$$

and

$$\Delta\omega = 0.05\pi$$

with

$$\omega_c = (\omega_p + \omega_s)/2 = 0.5\pi$$

The designed filter meets the given specifications although it has a much smaller ripple than the desired value.

The MATLAB program is provided as `ex10_12.m`.

Extension to Typical Frequency Selective Filter Design

For typical frequency selective filters, namely, **highpass**, **bandpass** and **bandstop** filters, inverse DTFT can be used

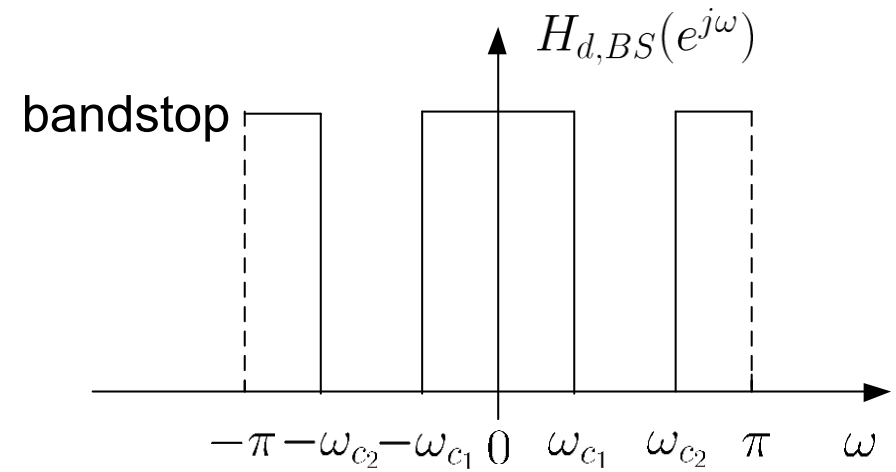
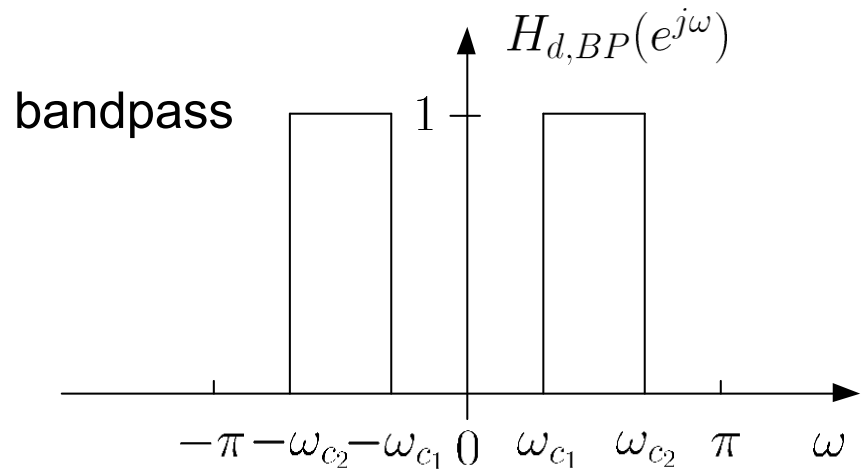
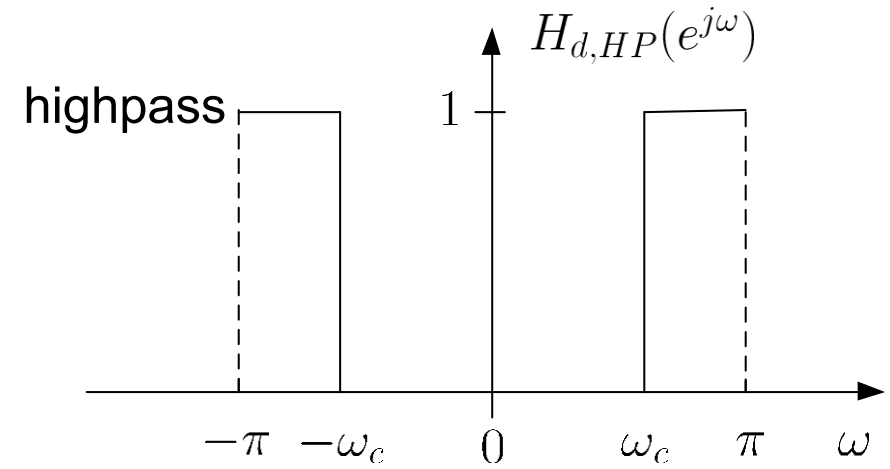
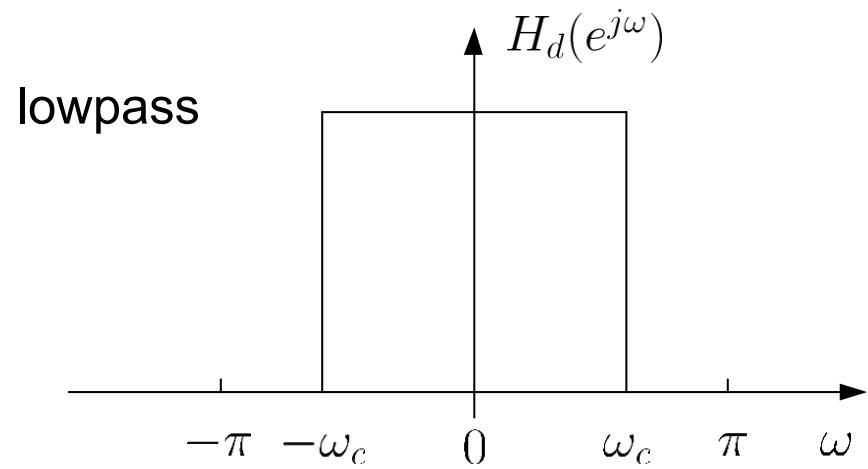


Fig.10.34: Typical frequency-selective filters

Nevertheless, we can utilize the lowpass filter to obtain impulse responses for $H_{d,HP}(e^{j\omega})$, $H_{d,BP}(e^{j\omega})$ and $H_{d,BS}(e^{j\omega})$

A highpass filter can be considered as subtraction of a lowpass filter from an allpass filter:

$$H_{d,HP}(e^{j\omega}) = 1 - H_d(e^{j\omega})$$
$$\Rightarrow h_{d,HP}[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{-\omega_c}^{\omega_c} e^{j\omega n} d\omega = \text{sinc}[n] - \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c n}{\pi}\right) \quad (10.24)$$

Assuming rectangular window:

$$h_{HP}[n] = \text{sinc}[n - \tau] - \frac{\omega_c}{\pi} \text{sinc}\left(\frac{\omega_c(n - \tau)}{\pi}\right) \quad (10.25)$$

where $\tau = (N - 1)/2$

Example 10.13

Use the window method with the rectangular window to find the impulse response of a linear-phase and causal FIR filter which approximates an ideal highpass filter whose cutoff frequency is $\omega_c = 0.5\pi$. It is required that the filter has a length of $N = 21$.

According to (10.25) with $N = 21$ and $\omega_c = 0.5\pi$:

$$h_{HP}[n] = \delta[n - 10] - 0.5\text{sinc}(0.5(n - 10)), \quad 0 \leq N \leq 20$$

Note that we can also use the MATLAB command `fir1(20, 0.5, 'high', boxcar(21), 'noscale')` to get the same result

The MATLAB program is provided as `ex10_13.m`.

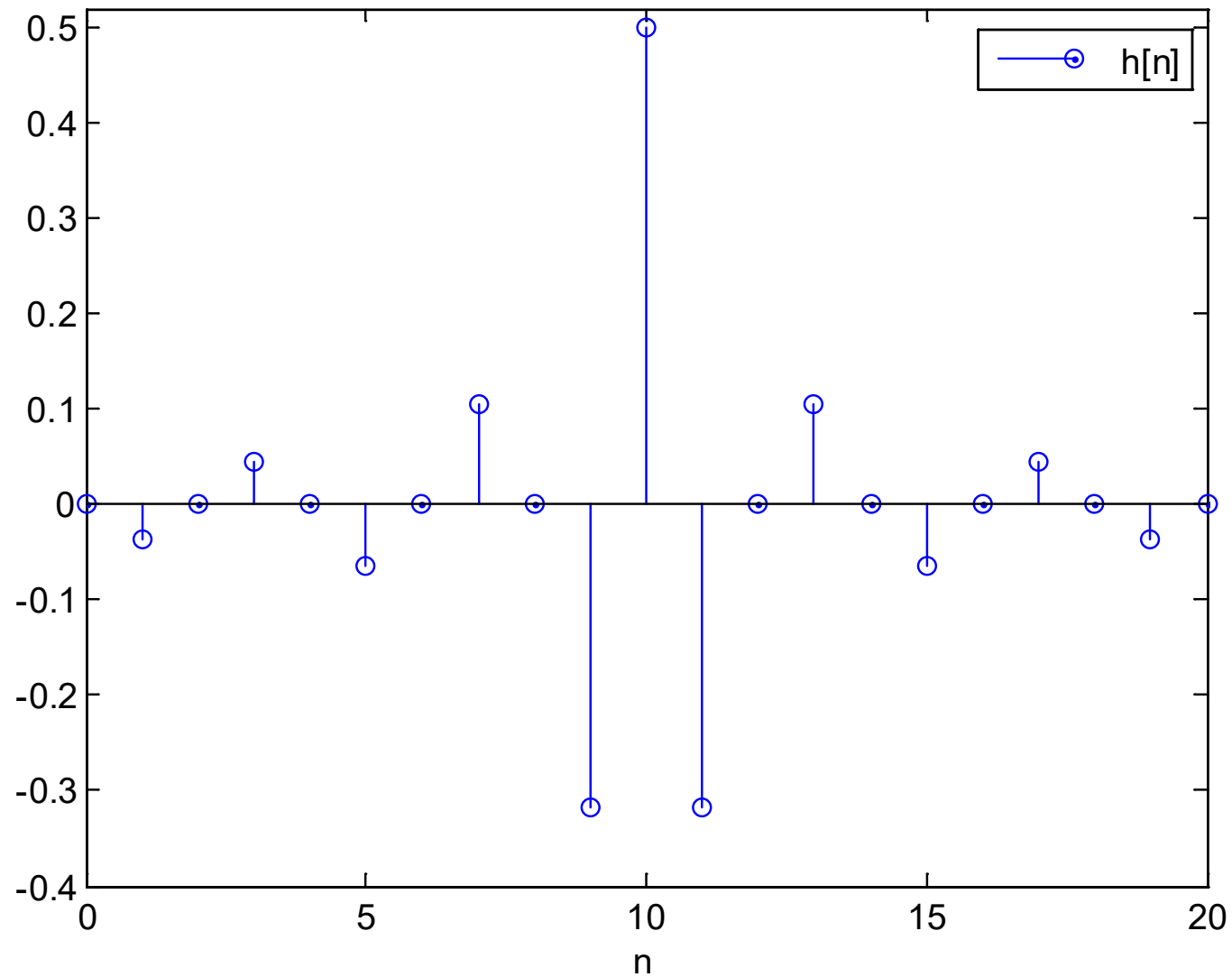


Fig.10.35: Impulse response of highpass filter with $N = 21$

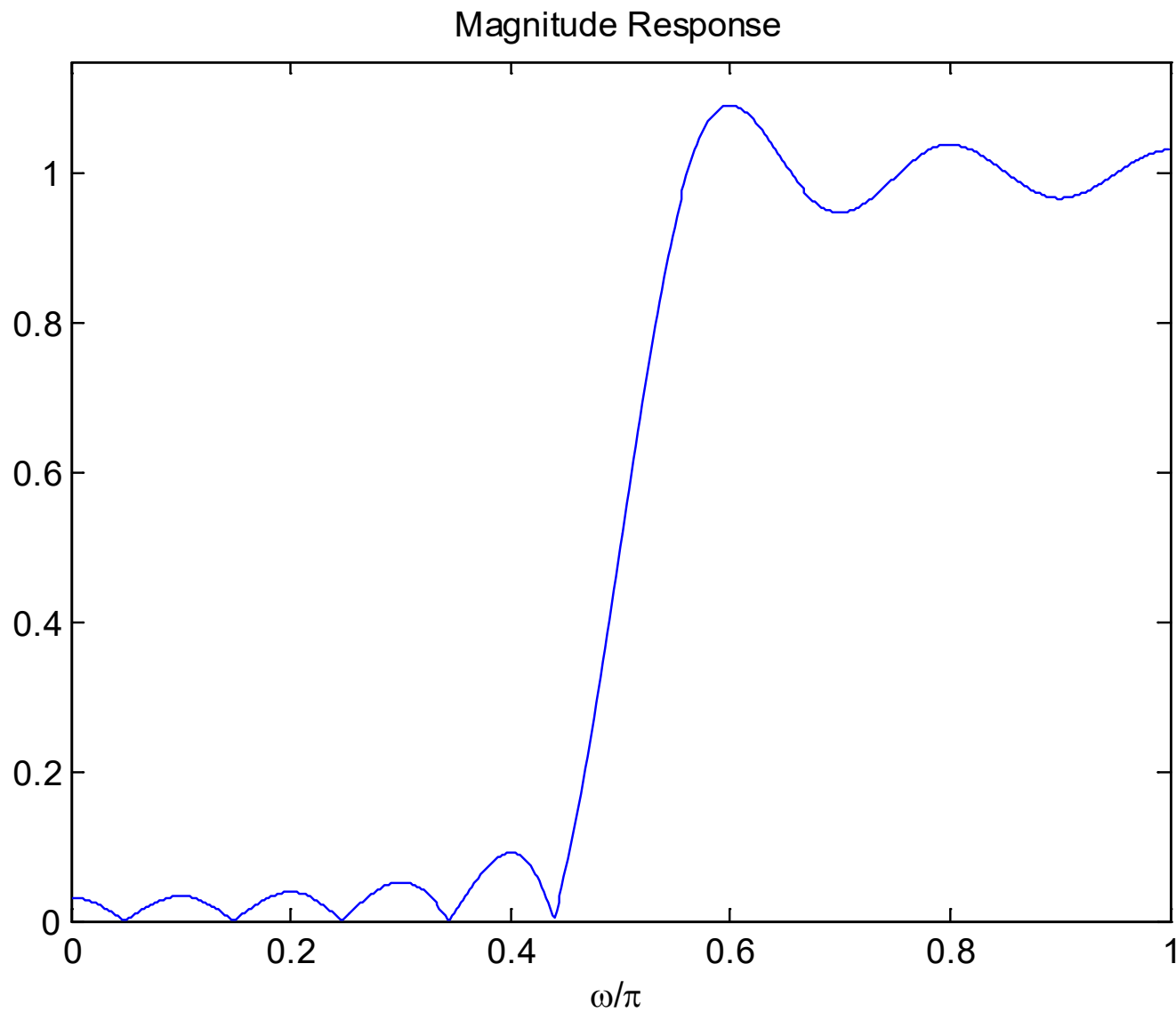


Fig.10.36: Magnitude response of highpass filter with $N = 21$

Similarly, an ideal bandpass filter can be designed from subtraction of two lowpass filters with cutoff frequencies ω_{c_1} and ω_{c_2} :

$$\begin{aligned} h_{d,BP}[n] &= \frac{1}{2\pi} \int_{-\omega_{c_2}}^{\omega_{c_2}} e^{j\omega n} d\omega - \frac{1}{2\pi} \int_{-\omega_{c_1}}^{\omega_{c_1}} e^{j\omega n} d\omega \\ &= \frac{\omega_{c_2}}{\pi} \text{sinc} \left(\frac{\omega_{c_2} n}{\pi} \right) - \frac{\omega_{c_1}}{\pi} \text{sinc} \left(\frac{\omega_{c_1} n}{\pi} \right) \end{aligned} \quad (10.26)$$

An ideal bandstop filter impulse response can be obtained by subtracting a bandpass filter from an allpass filter:

$$h_{d,BS}[n] = \delta[n] - \frac{\omega_{c_2}}{\pi} \text{sinc} \left(\frac{\omega_{c_2} n}{\pi} \right) + \frac{\omega_{c_1}}{\pi} \text{sinc} \left(\frac{\omega_{c_1} n}{\pi} \right) \quad (10.27)$$

Kaiser Window

A problem in typical window functions is that we cannot control the ripple δ

Kaiser window can control both δ and $\Delta\omega$ in a nearly optimal manner and it has the form of:

$$w[n] = \frac{I_0 \left(\beta \sqrt{1 - \left(1 - \frac{2n}{N-1} \right)^2} \right)}{I_0(\beta)}, \quad 0 \leq n \leq N-1 \quad (10.28)$$

where I_0 is the modified zero-order Bessel function. Apart from N , there is β which is responsible for the window shape

That is, by properly choosing N and β , the design specifications in terms of δ and $\Delta\omega$ can be precisely met

N is computed as:

$$N = \left\lceil \frac{-e_p - 8}{2.285\Delta\omega} \right\rceil + 1 \quad (10.29)$$

where $e_p = 20 \log_{10} \delta$ is the peak approximation error and $\lceil u \rceil$ rounds up u to the nearest integer

β is determined from:

$$\beta = \begin{cases} 0.1102(-e_p - 8.7), & -e_p > 50 \\ 0.5842(-e_p - 21)^{0.4} + 0.07886(-e_p - 21), & 21 \leq -e_p \leq 50 \\ 0, & -e_p < 21 \end{cases} \quad (10.30)$$

Example 10.14

Use the window method with the Kaiser window to design a linear-phase and causal FIR filter which approximates an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.5\pi$. The maximum allowable transition width is $\Delta\omega = 0.05\pi$ and the maximum allowable tolerance is $\delta = 0.005$.

Since

$$-e_p = -20 \log_{10}(0.005) = 46$$

and $\Delta\omega = 0.05\pi$, we then have:

$$N = \left\lceil \frac{46 - 8}{2.285 \cdot 0.05\pi} \right\rceil + 1 = 107$$

and

$$\beta = 0.5842(46 - 21)^{0.4} + 0.07886(46 - 21) = 4.09$$

As a result, the filter impulse response is:

$$h[n] = \frac{\sin(0.5\pi(n - 53))}{\pi(n - 53)} \cdot w[n], \quad 0 \leq n \leq 106$$

where $w[n]$ is determined from (10.28).

In summary:

$$e_p = -46 \text{ dB} \quad \text{or} \quad \delta \approx 0.005$$

$$\Delta\omega = 0.05\pi$$

with

$$\omega_c = (\omega_p + \omega_s)/2 = 0.5\pi$$

The designed filter **exactly** meets the given specifications

The MATLAB program is provided as `ex10_14.m`.

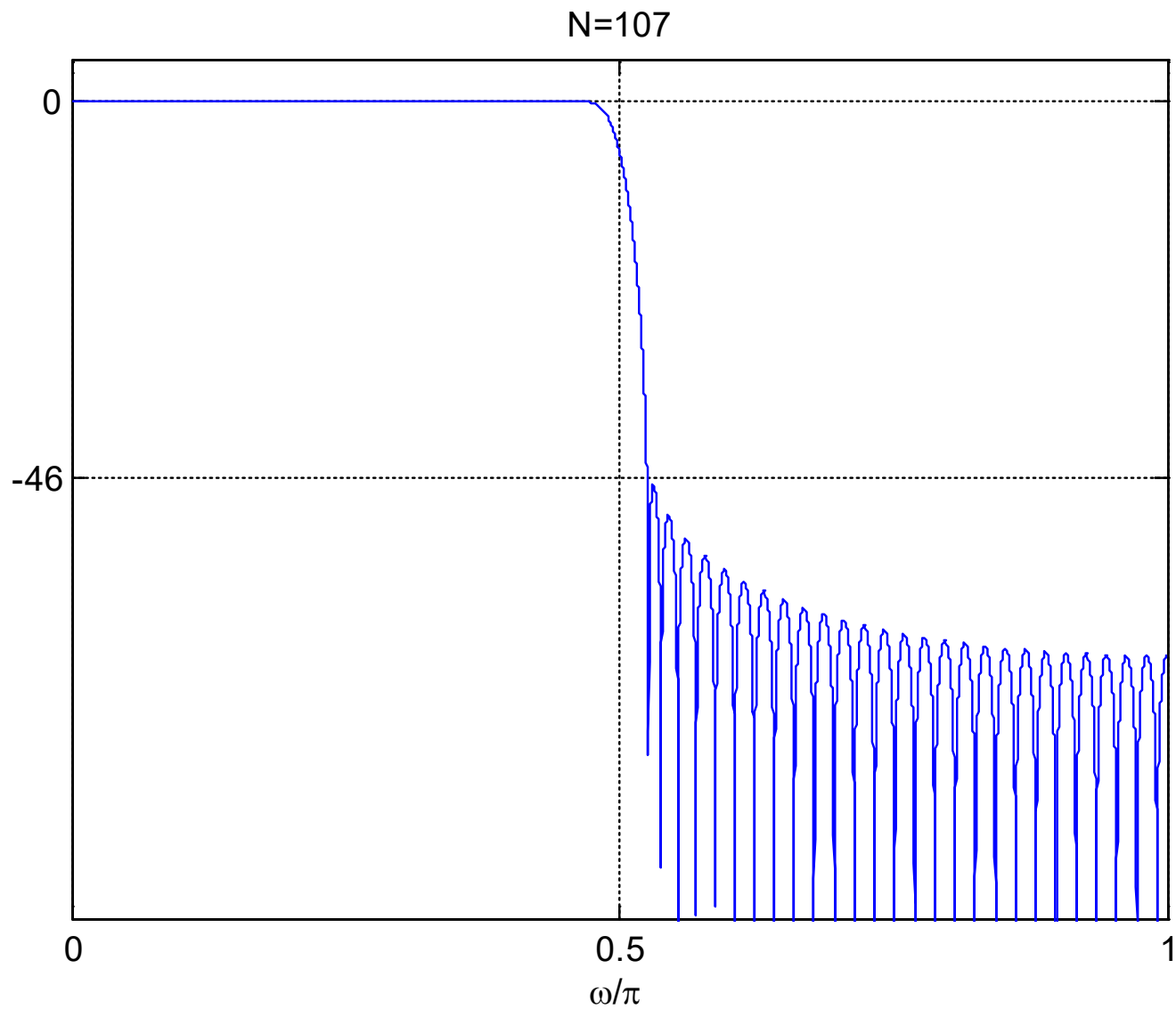


Fig.10.37: Magnitude response with Kaiser window

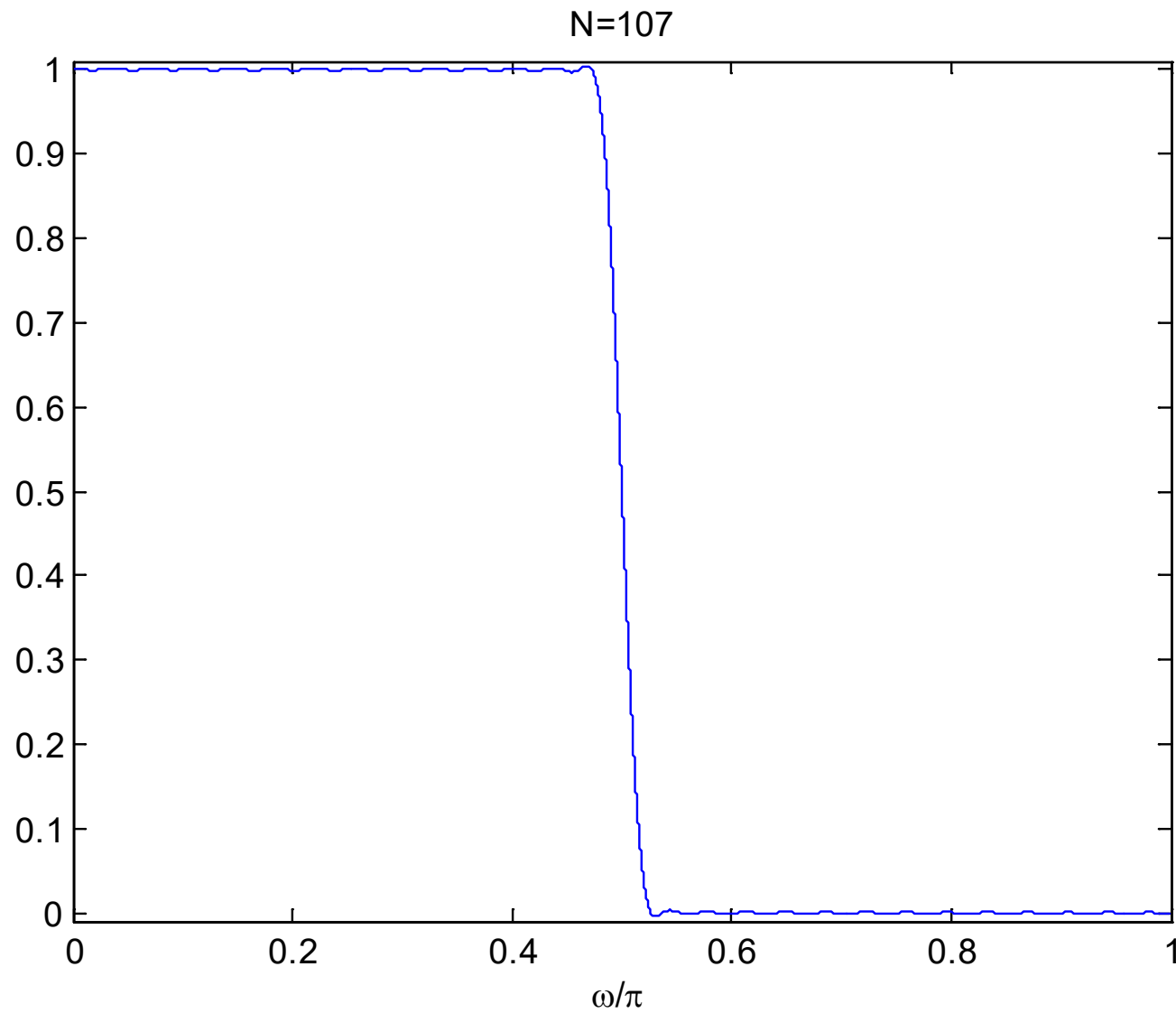


Fig.10.38: Amplitude response with Kaiser window

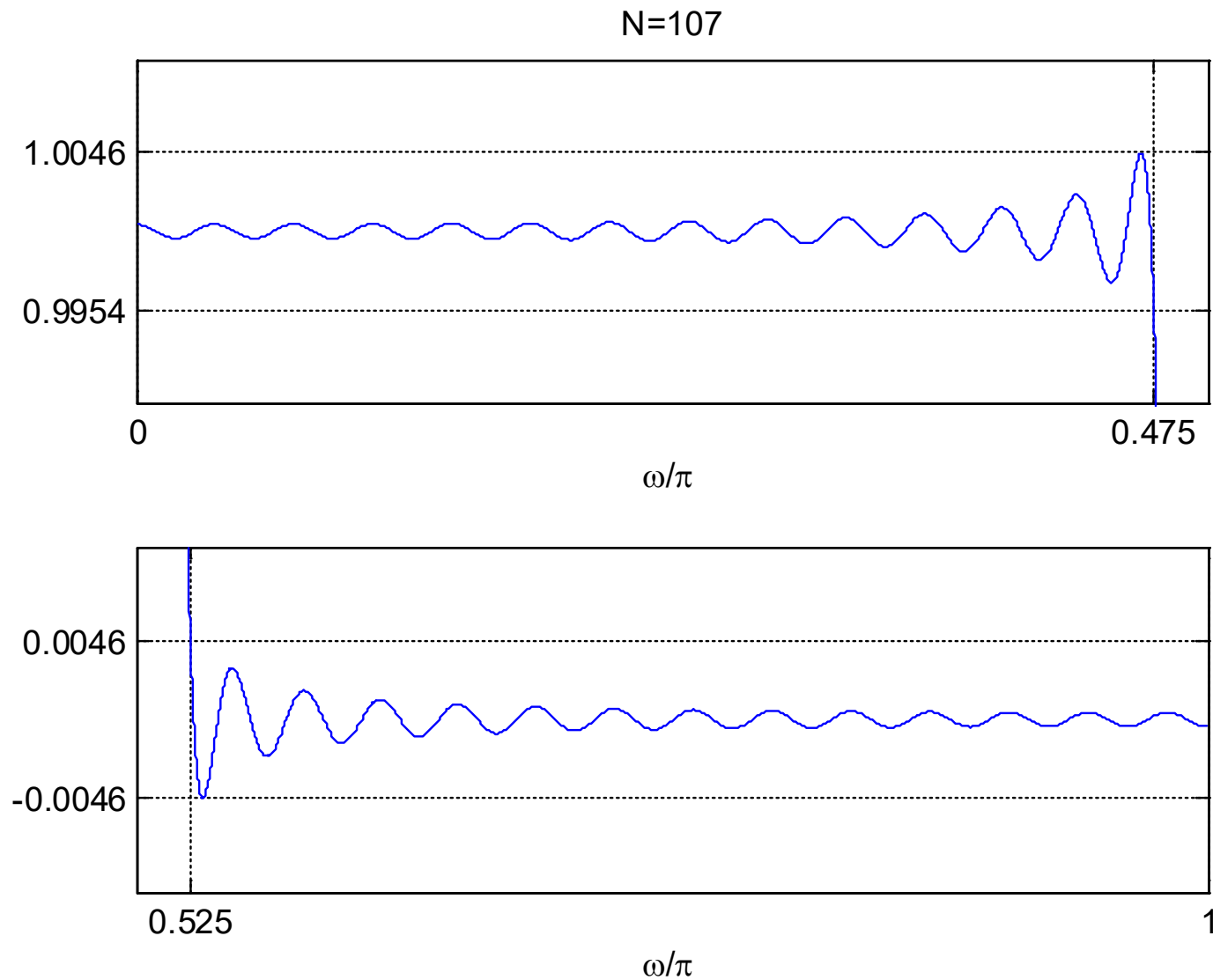


Fig.10.39: Zoomed amplitude responses with Kaiser window

Frequency Sampling Method

The basic idea is to utilize the discrete Fourier transform (DFT), which corresponds to samples of the desired frequency response $H_d(e^{j\omega})$, to produce $h[n]$

Recall:

$$H[k] = H_d(e^{j\omega})|_{\omega=\frac{2\pi k}{N}}, \quad k = 0, 1, \dots, N-1 \quad (10.31)$$

is equal to $H_d(e^{j\omega})$ sampled at N distinct frequencies between $\omega \in [0, 2\pi]$ with a uniform frequency spacing of $2\pi/N$

A causal and linear-phase filter is designed with 2 steps:

- Extract N uniformly-spaced samples from $H_d(e^{j\omega})$ in the frequency range of $\omega \in [0, 2\pi]$
- Compute $h[n]$ by taking the inverse DFT of $H[k]$:

$$h[n] = \frac{1}{N} \sum_{k=0}^{N-1} H[k] W_N^{-kn}, \quad W_N = e^{-j2\pi/N}, \quad 0 \leq n \leq N-1 \quad (10.32)$$

Taking z transform of $h[n]$:

$$\begin{aligned} H(z) &= \sum_{n=0}^{N-1} h[n] z^{-n} \\ &= \sum_{n=0}^{N-1} \left(\frac{1}{N} \sum_{k=0}^{N-1} H[k] e^{j2\pi kn/N} \right) z^{-n} \\ &= \frac{1}{N} \sum_{k=0}^{N-1} H[k] \sum_{n=0}^{N-1} \left(e^{j(2\pi/N)k} z^{-1} \right)^n \\ &= \frac{1 - z^{-N}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{j(2\pi/N)k} z^{-1}} \end{aligned} \quad (10.33)$$

The filter frequency response is then:

$$H(e^{j\omega}) = \frac{1 - e^{-j\omega N}}{N} \sum_{k=0}^{N-1} \frac{H[k]}{1 - e^{j(2\pi/N)k} e^{-j\omega}} \quad (10.34)$$

From (10.33) and (7.7) together with (10.31):

$$H(e^{j2\pi r/N}) = \frac{1}{N} \sum_{k=0}^{N-1} H[k] \sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n} = H[r]$$

$$\Rightarrow H(e^{j\omega})|_{\omega=\frac{2\pi k}{N}} = H[k] = H_d(e^{j\omega})|_{\omega=\frac{2\pi k}{N}}, \quad k = 0, 1, \dots, N-1 \quad (10.35)$$

It is simple as only uniformly-spaced samples of the desired frequency response or the DFT coefficients are needed

At $\omega = (2\pi/N)k$, $k = 0, 1, \dots, N-1$, $H(e^{j\omega})$ equals $H_d(e^{j\omega})$ but we cannot control the values of the remaining frequency points

It lacks flexibility in specifying the passband and stopband cutoff frequencies since placement of “1” & “0” & transition samples is constrained to integer multiples of $2\pi/N$

Example 10.15

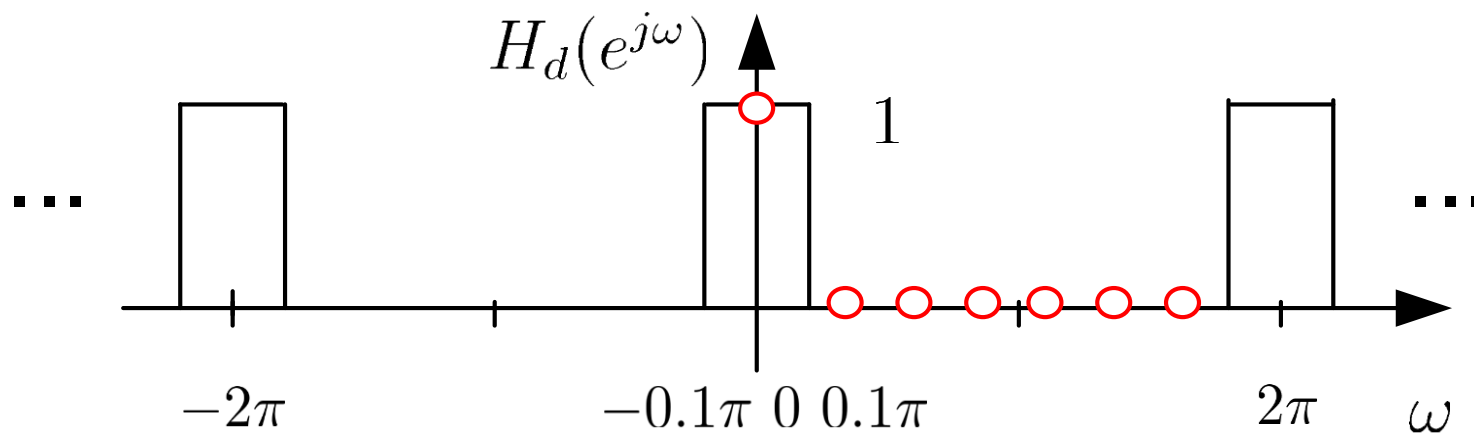
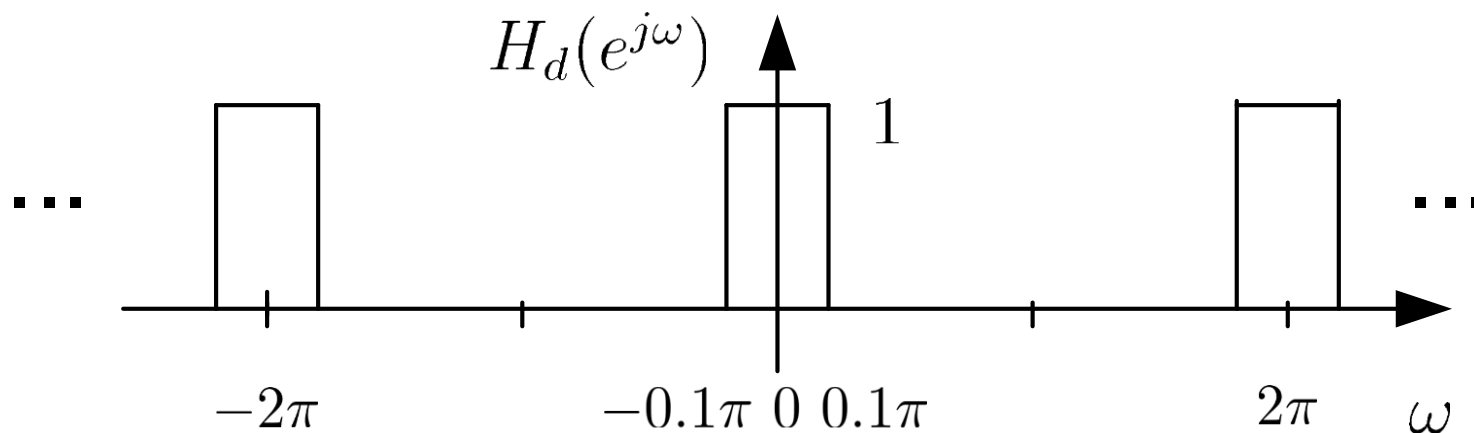
Use the frequency sampling method to design a linear-phase and causal FIR filter with a length of $N = 7$ to approximate an ideal lowpass filter whose cutoff frequency is $\omega_c = 0.1\pi$.

From Example 10.7, the ideal frequency response with linear phase is

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\omega\tau}, & |\omega| < \omega_c \\ 0, & -\pi < \omega < -\omega_c, \quad \omega_c < \omega < \pi \end{cases}$$

where $\tau = (N - 1)/2 = 3$ and $\omega_c = 0.1\pi$

Note that DFT corresponds to the interval $\omega \in [0, 2\pi]$



We consider the frequency interval of $\omega \in [0, 2\pi]$:

$$H_d(e^{j\omega}) = \begin{cases} 1 \cdot e^{-j\omega\tau}, & 0 < \omega < \omega_c \\ 0, & \omega_c < \omega < 2\pi - \omega_c \\ 1 \cdot e^{-j(\omega-2\pi)\tau}, & 2\pi - \omega_c < \omega < 2\pi \end{cases}$$

Extracting the values of $H_d(e^{j\omega})$ at $\omega = (2\pi/7)k$, $k = 0, 1, \dots, 6$:

$$H[k] = \begin{cases} 1, & k = 0 \\ 0, & k = 1, 2, \dots, 6 \end{cases}$$

Taking the inverse DFT:

$$h[n] = \frac{1}{7}, \quad n = 0, 1, \dots, 6$$

The MATLAB program is provided as `ex10_15.m`.

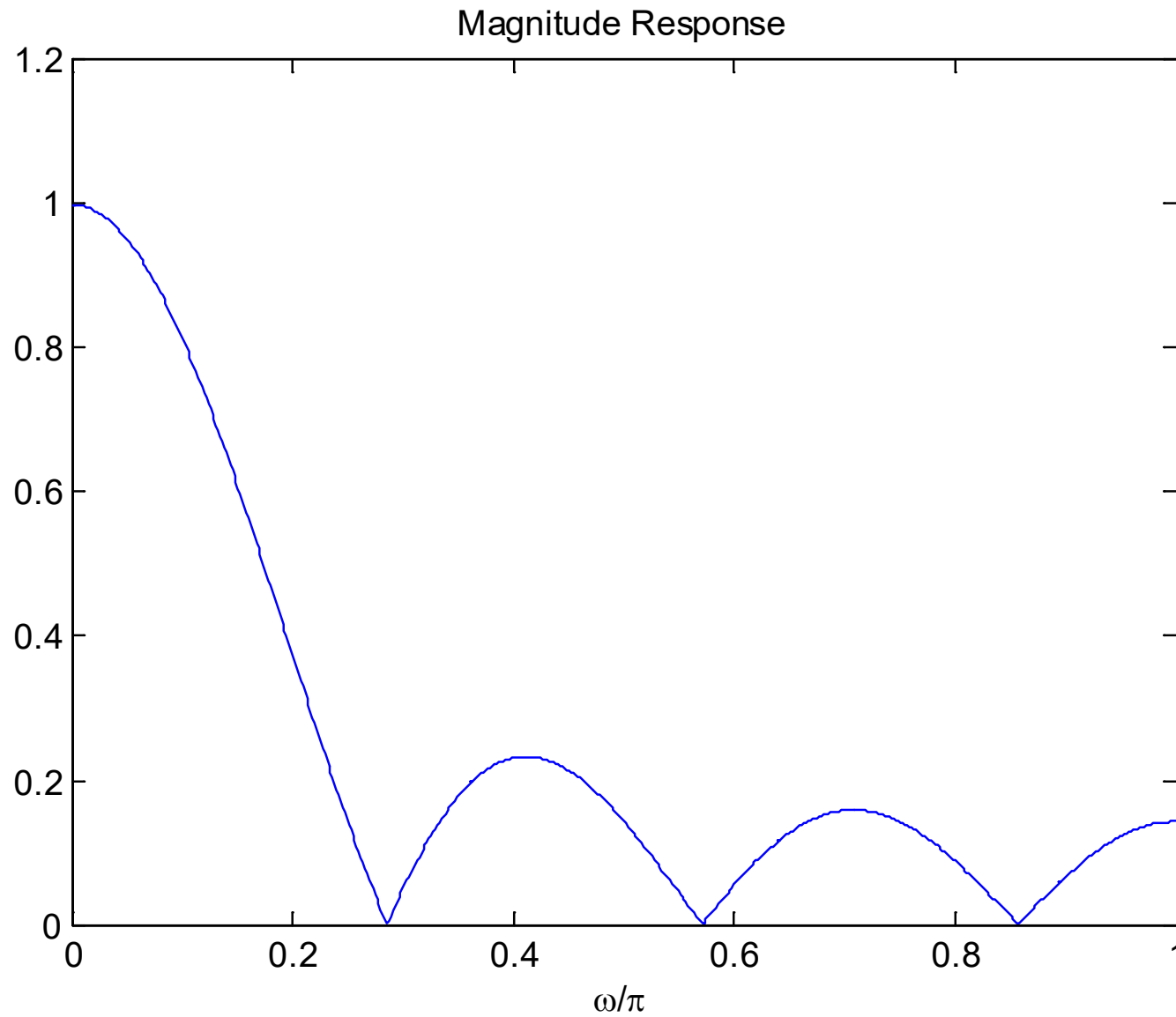


Fig.10.40: Magnitude response based on frequency sampling

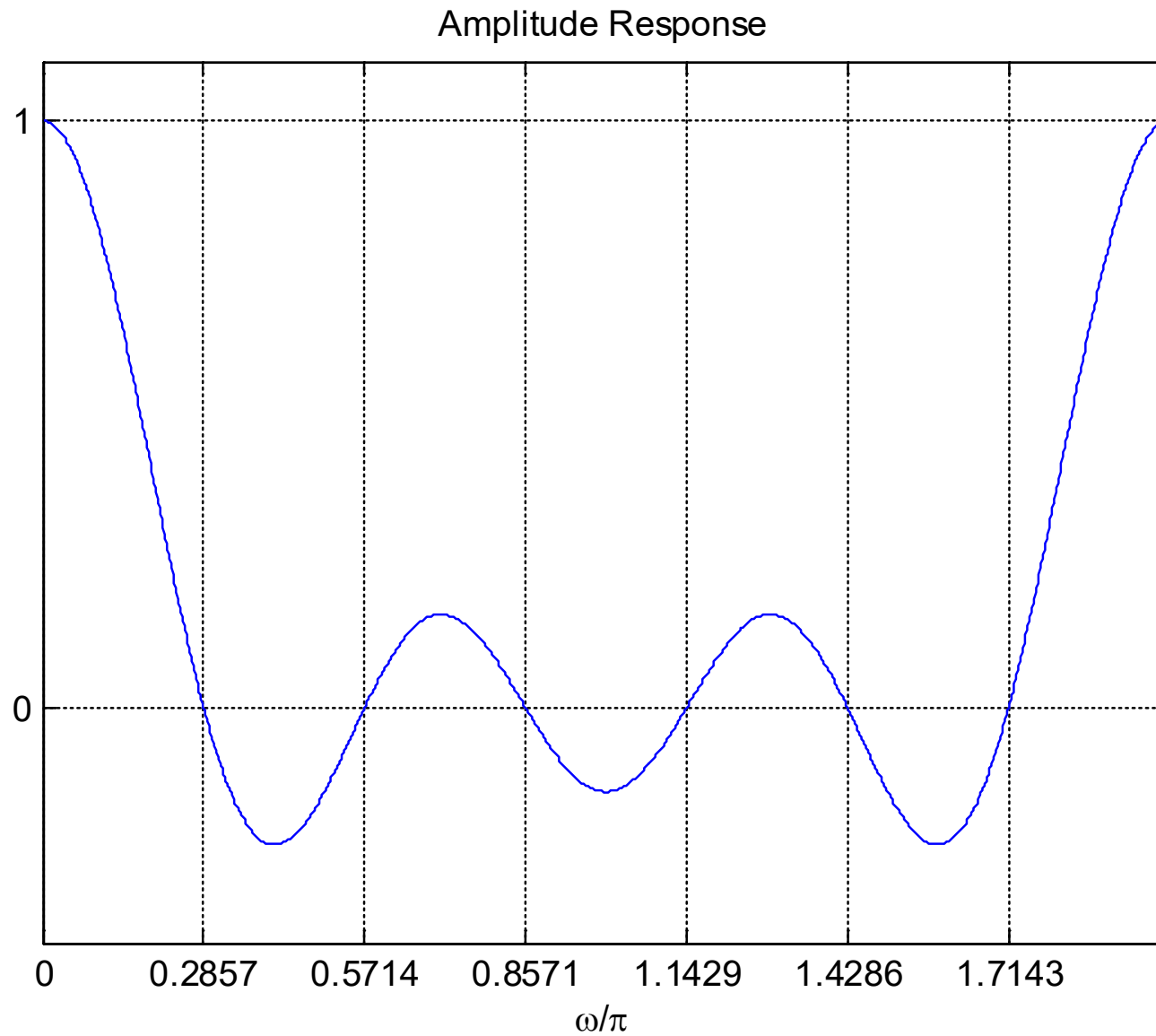


Fig.10.41: Amplitude response based on frequency sampling

Optimal Equiripple Method

The basic idea is to **evenly distribute** the ripples in both passband and stopband

- Required filter length will be **shorter** than that of the window method where its $H(e^{j\omega})$ exactly meets the passband or stopband ripple specification at **one** frequency and is superior to it at other frequencies in the band
- Allow $\delta_1 \neq \delta_2$
- Passband and stopband frequencies can be precisely specified although $\omega_p = \omega_c - \Delta\omega/2$ and $\omega_s = \omega_c + \Delta\omega/2$ are implicitly implied in the window method

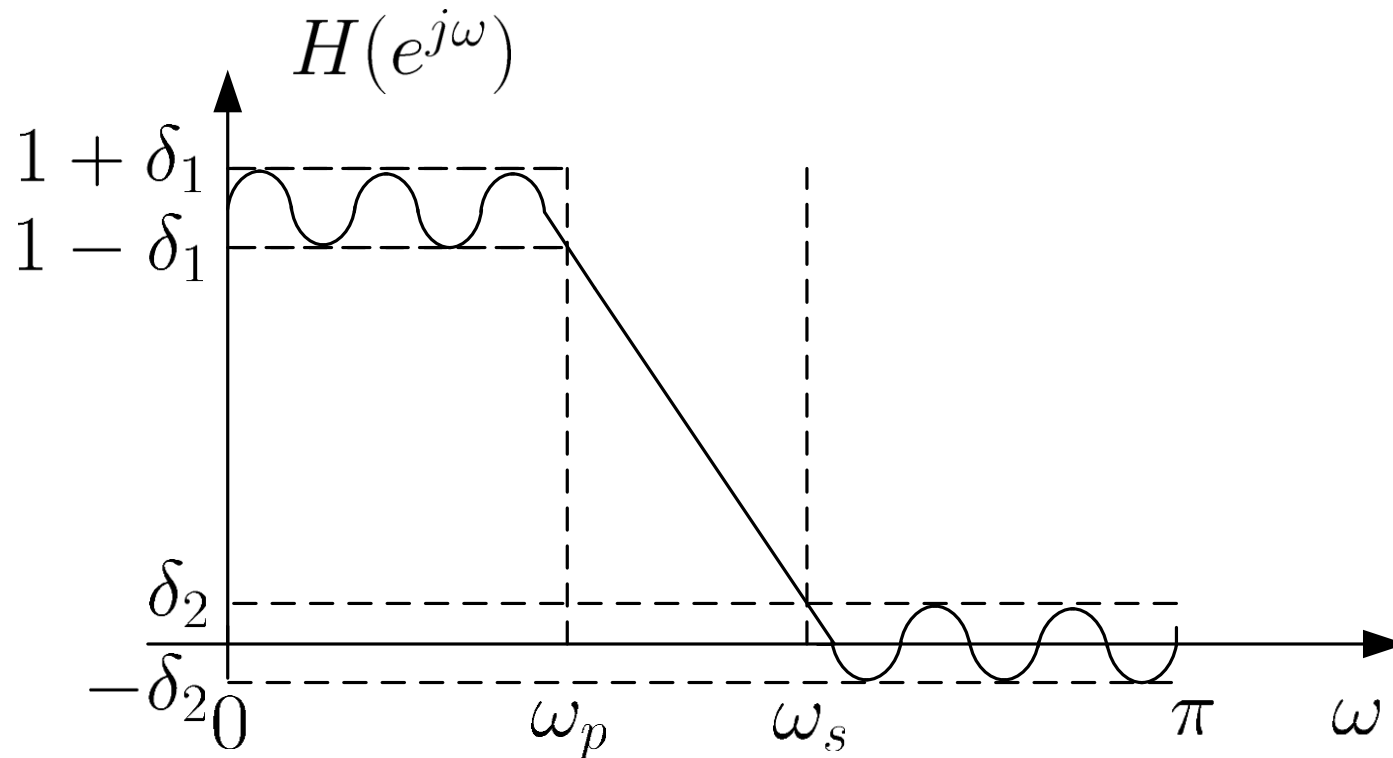


Fig.10.42: Illustration of optimal equiripple lowpass filter

Ripples are uniformly distributed such that $H(e^{j\omega})$ reaches its maximum deviations of δ_1 and δ_2 **more than once**

The impulse response $h[n]$ of optimal equiripple design is determined from:

$$h[n] = \min_{\{\tilde{h}[n]\}} \left(\max_{0 \leq \omega \leq \omega_p, \omega_s \leq \omega \leq \pi} |E(e^{j\omega})| \right), \quad n = 0, 1, \dots, N-1 \quad (10.36)$$

where

$$E(e^{j\omega}) = W(e^{j\omega})(H_d(e^{j\omega}) - \tilde{H}(e^{j\omega})) \quad (10.37)$$

which corresponds to a minmax optimization problem

$E(e^{j\omega})$ is the frequency-domain error between the desired and actual responses weighted by $W(e^{j\omega})$

$W(e^{j\omega})$ is the weighting function incorporates all specification parameters, namely, δ_1 , δ_2 , ω_p and ω_s , into the design process

For example, in lowpass filter design, $W(e^{j\omega})$ has the form of:

$$W(e^{j\omega}) = \begin{cases} \frac{\delta_2}{\delta_1}, & 0 \leq \omega \leq \omega_p \\ 1, & \omega_s \leq \omega \leq \pi \end{cases} \quad (10.38)$$

When $\delta_1 > \delta_2$, there is a larger weighting at the stopband. On the other hand, $\delta_1 < \delta_2$ implies a larger weighting at the passband

To solve for the minmax problem of (10.36), we can make use of the Parks-McClellan algorithm which requires iterations. The corresponding MATLAB command is `firpm` where the filter length parameter N is also required.

We can employ

$$N = \left\lceil \frac{-10 \log_{10}(\delta_1 \delta_2) - 13}{2.324(\omega_s - \omega_p)} \right\rceil \quad (10.39)$$

to get its initial estimate and then compute $h[n]$. If the tolerance specifications are not met, we increment N until the maximum deviations are bounded by δ_1 and δ_2 .

Example 10.16

Use the optimal equiripple method to design a linear-phase and causal FIR filter which approximates an ideal lowpass filter whose passband frequency is $\omega_p = 0.475\pi$ and stopband frequency is $\omega_s = 0.525\pi$. The maximum allowable tolerance is $\delta = \delta_1 = \delta_2 = 0.005$ in both passband and stopband.

According to (10.39), an initial value of N is computed as:

$$N = \left\lceil \frac{-10 \log_{10}(0.005 \cdot 0.005) - 13}{2.324(0.525\pi - 0.425\pi)} \right\rceil = 91$$

Starting with $N = 91$ in the Parks-McClellan algorithm, we increment its value until $N = 96$ so that the tolerance specifications are met.

The MATLAB program is provided as `ex10_16.m`.

In summary:

$$e_p = -46 \text{ dB} \quad \text{or} \quad \delta_1 = \delta_2 = 0.005$$

$$\omega_p = 0.475\pi, \quad \omega_s = 0.525\pi$$

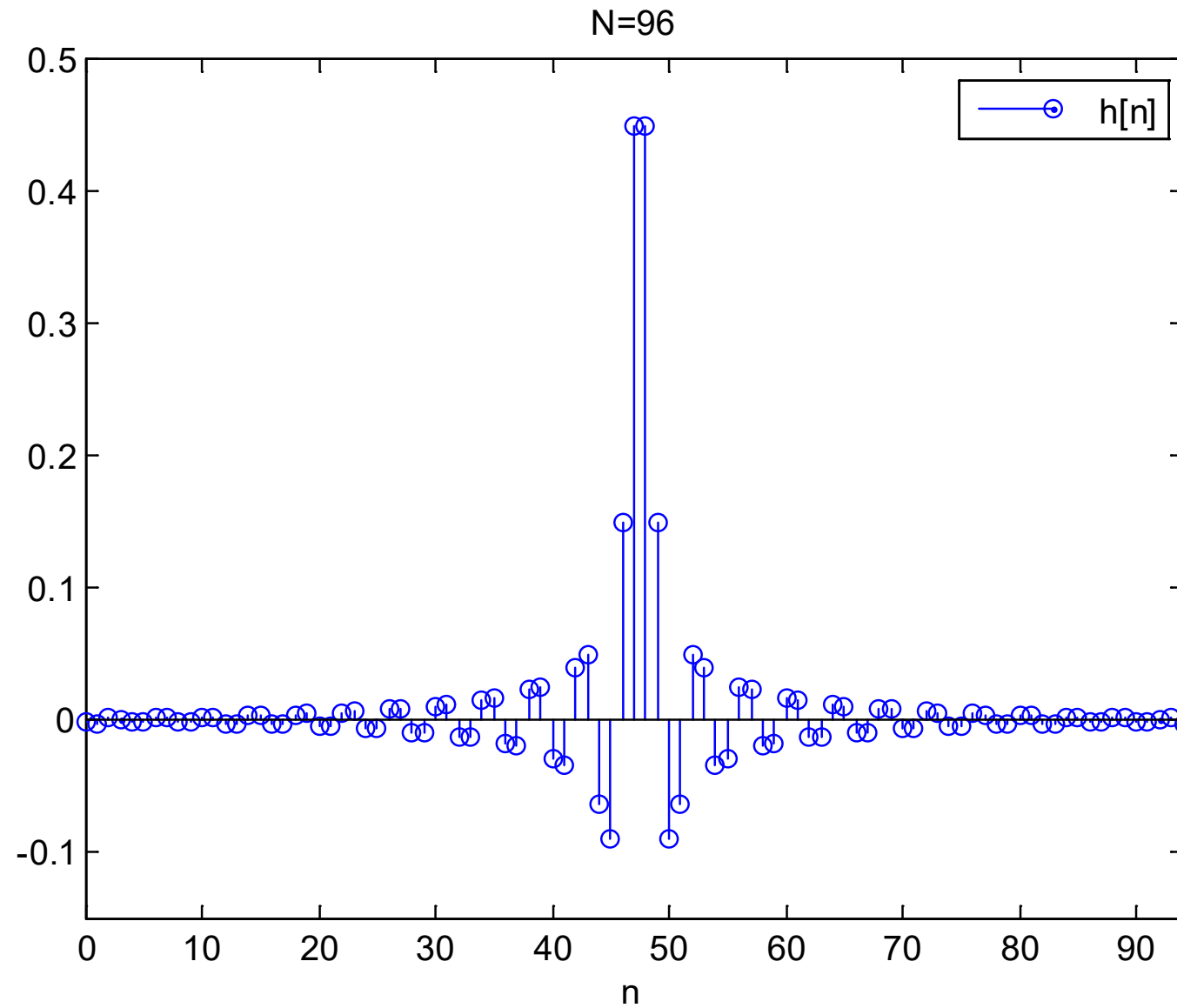


Fig.10.43: Impulse response of optimal equiripple lowpass filter

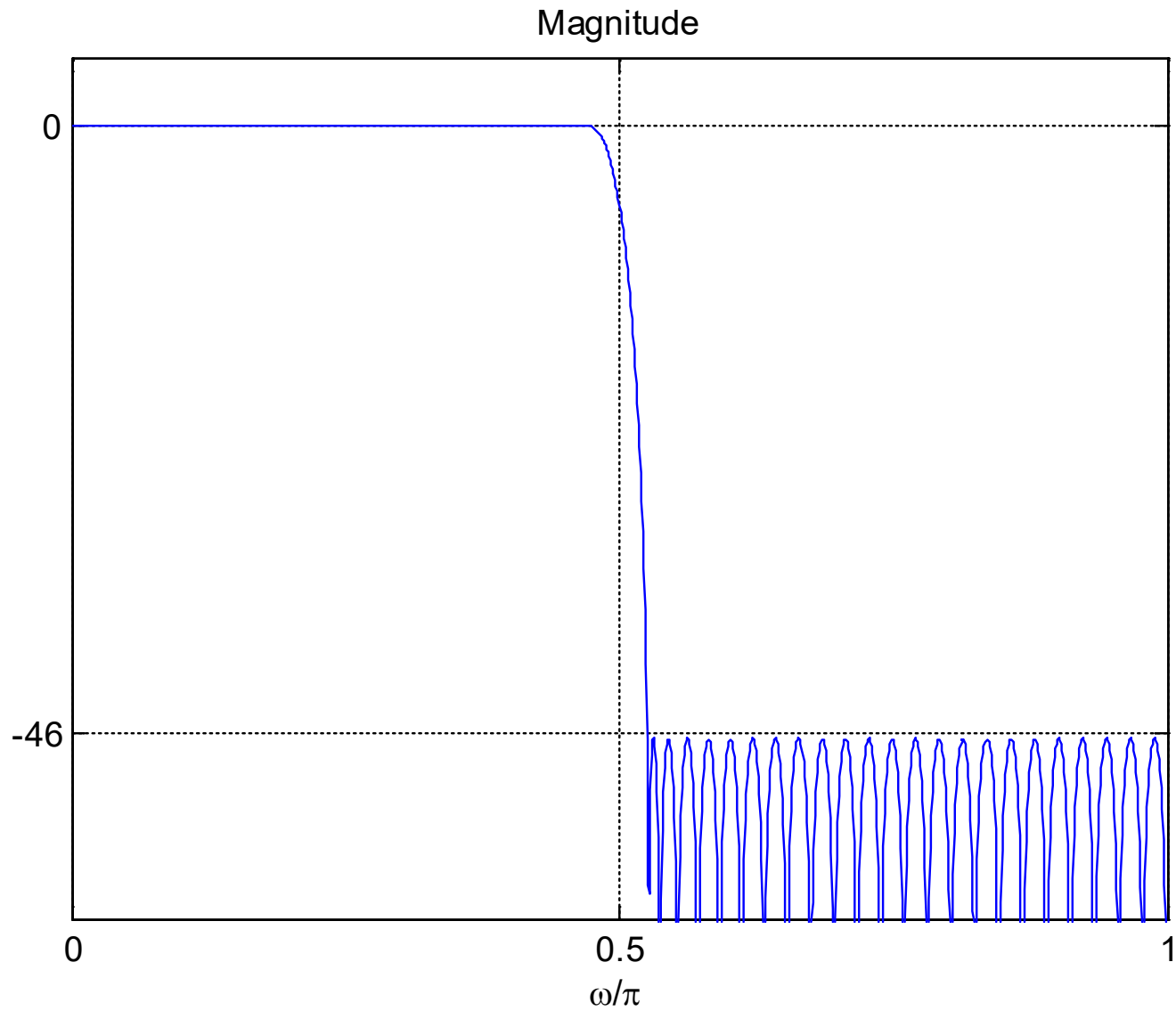


Fig.10.44: Magnitude response of optimal equiripple lowpass filter

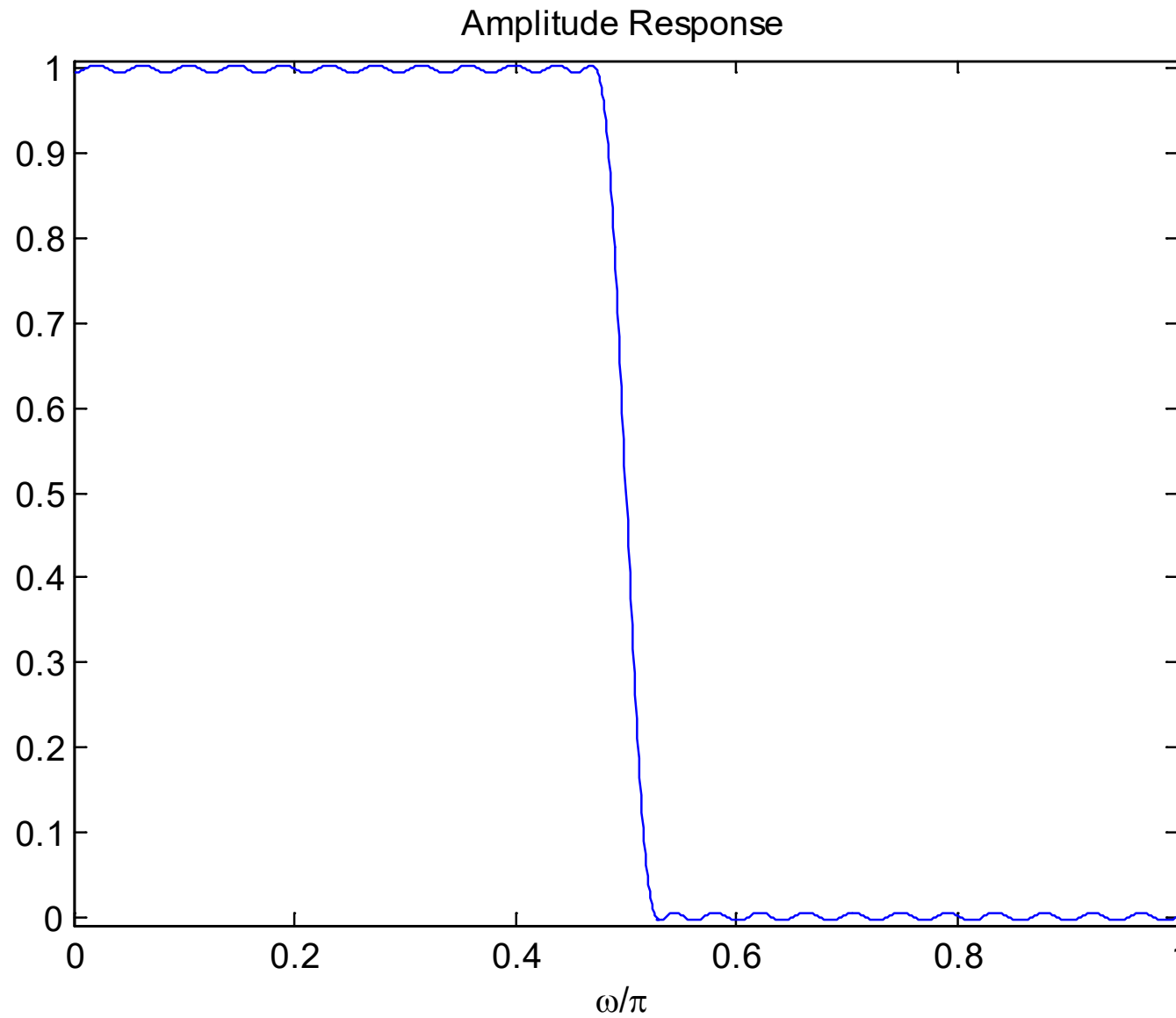


Fig.10.45: Amplitude response of optimal equiripple lowpass filter

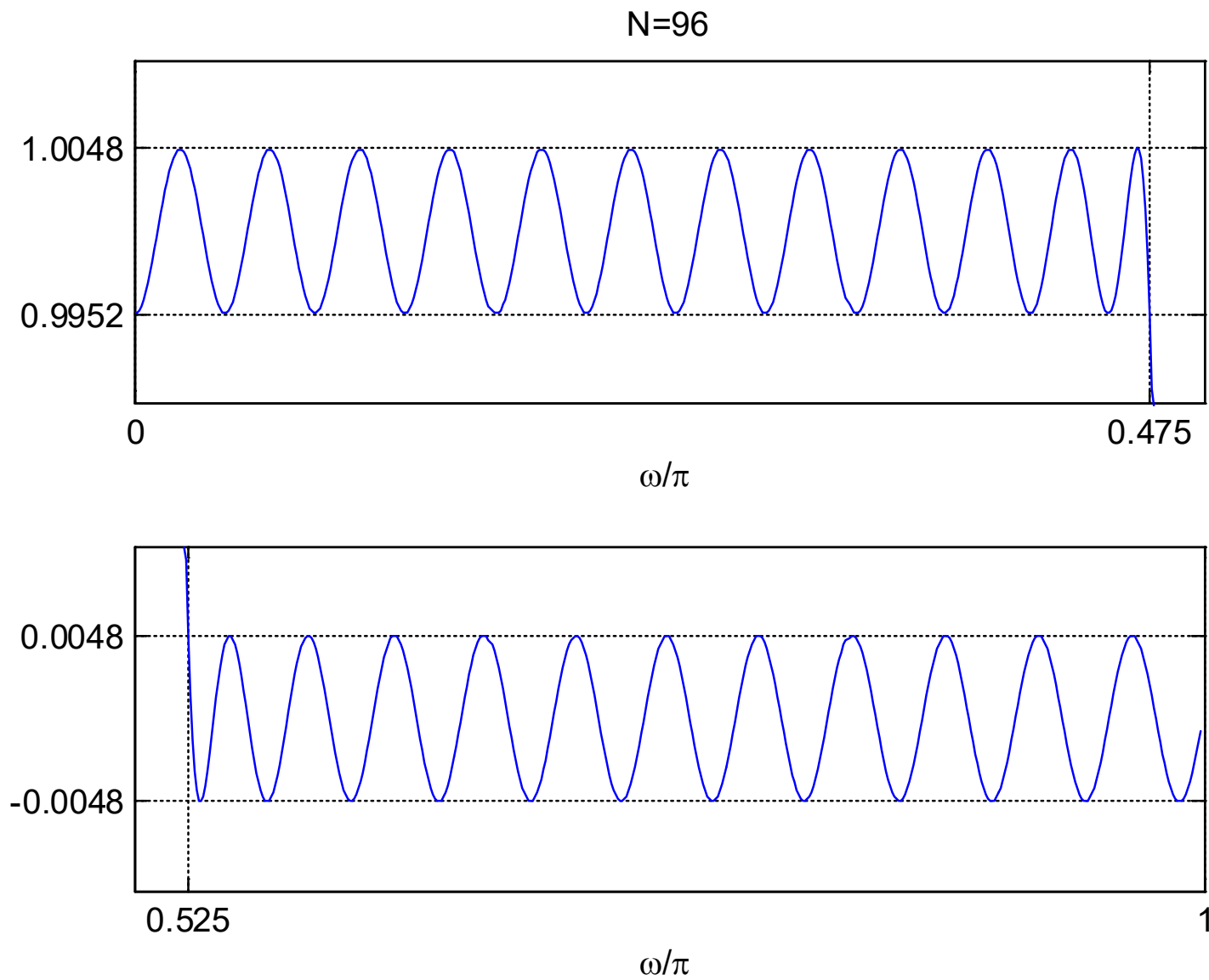


Fig.10.46: Zoomed amplitude responses of optimal equiripple lowpass filter