Review of Analog Signal Analysis

Chapter Intended Learning Outcomes:

(i) Review of Fourier series which is used to analyze continuous-time periodic signals

(ii) Review of Fourier transform which is used to analyze continuous-time aperiodic signals

(iii) Review of analog linear time-invariant system

Fourier series and Fourier transform are the tools for analyzing analog signals. Basically, they are used for signal conversion between time and frequency domains:

$$x(t) \leftrightarrow X(j\Omega) \tag{2.1}$$

Fourier Series

- For analysis of continuous-time periodic signals
- Express periodic signals using harmonically related sinusoids with frequencies $\cdots -\Omega_0, 0, \Omega_0, 2\Omega_0, \cdots$ where Ω_0 is called the fundamental frequency or first harmonic, $2\Omega_0$ is called the second harmonic, $3\Omega_0$ is called the third harmonic, and so on
- In the frequency domain, Ω only takes discrete values at $\cdots -\Omega_0, 0, \Omega_0, 2\Omega_0, \cdots$

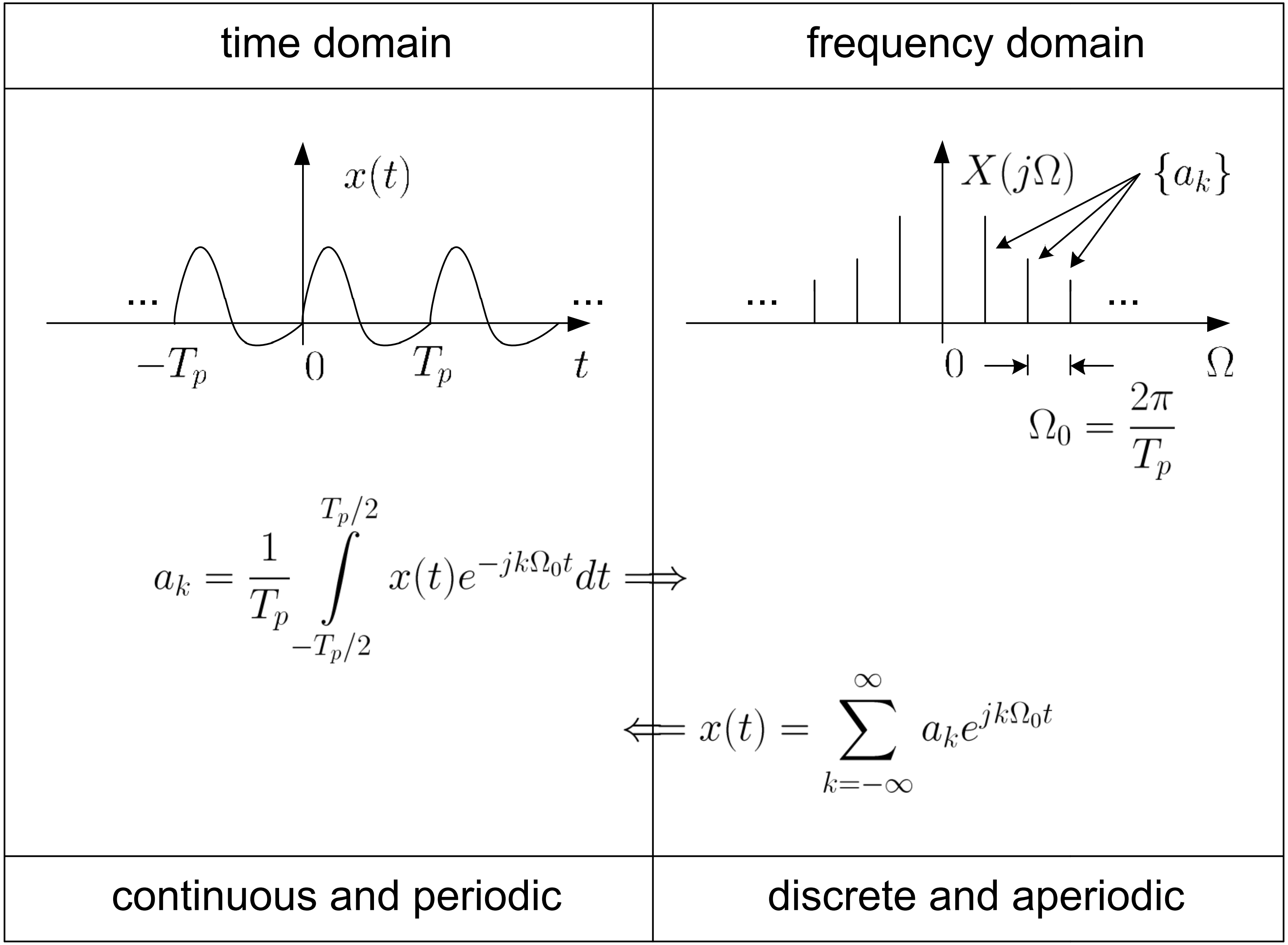


Fig.2.1: Illustration of Fourier series

A continuous-time function $x(t)$ is said to be periodic if there exists $T_p > 0$ such that

$$x(t) = x(t + T_p), \qquad t \in (-\infty, \infty) \tag{2.2}$$

The smallest T_p for which (2.2) holds is called the fundamental period

The fundamental frequency is related to T_p as:

$$\Omega_0 = \frac{2\pi}{T_p} \tag{2.3}$$

Every periodic function can be expanded into a Fourier series as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{2.4}$$

where

$$a_k = \frac{1}{T_p} \int_{-T_p/2}^{T_p/2} x(t) e^{-jk\Omega_0 t} dt, \qquad k = \cdots -1, 0, 1, 2, \cdots \tag{2.5}$$

are called Fourier series coefficients

As $X(j\Omega)$ is characterized by $\{a_k\}$, the Fourier series coefficients in fact correspond to the frequency representation of $x(t)$

Generally, a_k is complex and we can also use magnitude and phase for its representation

$$|a_k| = \sqrt{(\Re\{a_k\})^2 + (\Im\{a_k\})^2} \tag{2.6}$$

and

$$\angle(a_k) = \tan^{-1}\left(\frac{\Im\{a_k\}}{\Re\{a_k\}}\right) \tag{2.7}$$

<u>Example 2.1</u>

Find the Fourier series coefficients for $x(t) = \cos(10\pi t) + \cos(20\pi t)$

It is clear that the fundamental frequency of $x(t)$ is $\Omega_0 = 10\pi$. According to (2.3), the fundamental period is thus equal to $T_p = 2\pi/\Omega_0 = 1/5$, which is validated as follows:

$$\begin{aligned} x\left(t + \frac{1}{5}\right) &= \cos\left(10\pi\left(t + \frac{1}{5}\right)\right) + \cos\left(20\pi\left(t + \frac{1}{5}\right)\right) \\ &= \cos(10\pi t + 2\pi) + \cos(20\pi t + 4\pi) \\ &= \cos(10\pi t) + \cos(20\pi t) \end{aligned}$$

With the use of Euler formulas:

$$\cos(u) = \frac{e^{ju} + e^{-ju}}{2}$$

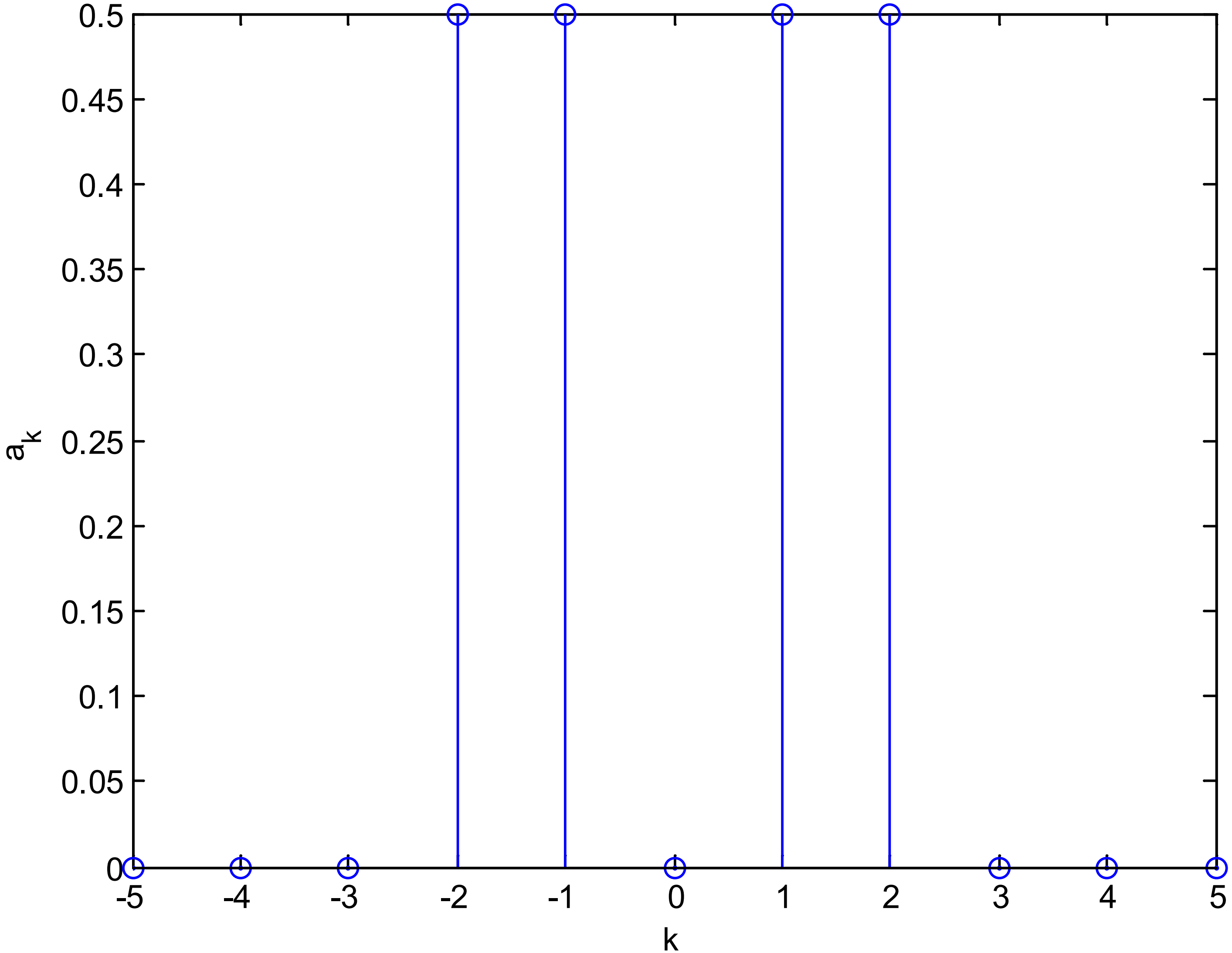
and

$$\sin(u) = \frac{e^{ju} - e^{-ju}}{2j}$$

we can express $x(t)$ as:

$$\begin{aligned} x(t) &= \cos(10\pi t) + \cos(20\pi t) \\ &= \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2} + \frac{e^{j2\Omega_0 t} + e^{-j2\Omega_0 t}}{2} \\ &= \frac{1}{2}e^{-j2\Omega_0 t} + \frac{1}{2}e^{-j\Omega_0 t} + \frac{1}{2}e^{j\Omega_0 t} + \frac{1}{2}e^{j2\Omega_0 t} \end{aligned}$$

By inspection and using (2.4), we have $a_{-2} = a_{-1} = a_1 = a_2 = 1/2$ while all other Fourier series coefficients are equal to zero



Can we use (2.5)? Why?

Example 2.2

Find the Fourier series coefficients for $x(t) = 1 + \sin(\Omega_0 t) + 2\cos(\Omega_0 t) + \cos(3\Omega_0 t + \pi/4)$.

With the use of Euler formulas, $x(t)$ can be written as:

$$
\begin{aligned}
x(t) &= 1 + \left(1 + \frac{1}{2j}\right) e^{j\Omega_0 t} + \left(1 - \frac{1}{2j}\right) e^{-j\Omega_0 t} + \frac{1}{2} e^{j\pi/4} e^{3j\Omega_0 t} + \frac{1}{2} e^{-j\pi/4} e^{-3j\Omega_0 t} \\
&= \frac{\sqrt{2}}{4}(1-j)e^{-3j\Omega_0 t} + \left(1 + j\frac{1}{2}\right) e^{-j\Omega_0 t} + 1 + \left(1 - j\frac{1}{2}\right) e^{j\Omega_0 t} \\
&\quad + \frac{\sqrt{2}}{4}(1+j)e^{3j\Omega_0 t}
\end{aligned}
$$

Using (2.4), we have:

$$a_k = \begin{cases} \frac{\sqrt{2}}{4}(1-j), & k=-3 \\ 1+\frac{j}{2}, & k=-1 \\ 1, & k=0 \\ 1-\frac{j}{2}, & k=1 \\ \frac{\sqrt{2}}{4}(1+j), & k=3 \\ 0, & \text{otherwise} \end{cases}$$

To plot $\{a_k\}$, we need to compute $|a_k|$ and $\angle(a_k)$ for all k, e.g.,

$$|a_{-3}| = \sqrt{\left(\frac{\sqrt{2}}{4}\right)^2 + \left(-\frac{\sqrt{2}}{4}\right)^2} = \frac{1}{2}$$

and

$$\angle(a_{-3}) = \tan^{-1}(-1) = -\frac{\pi}{4}$$

Example 2.3

Find the Fourier series coefficients for $x(t)$, which is a periodic continuous-time signal of fundamental period T and is a pulse with a width of $2T_0$ in each period. Over the specific period from $-T/2$ to $T/2$, $x(t)$ is:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

with $T > 2T_0$.

Why we need the condition regarding *T* and the pulse width?

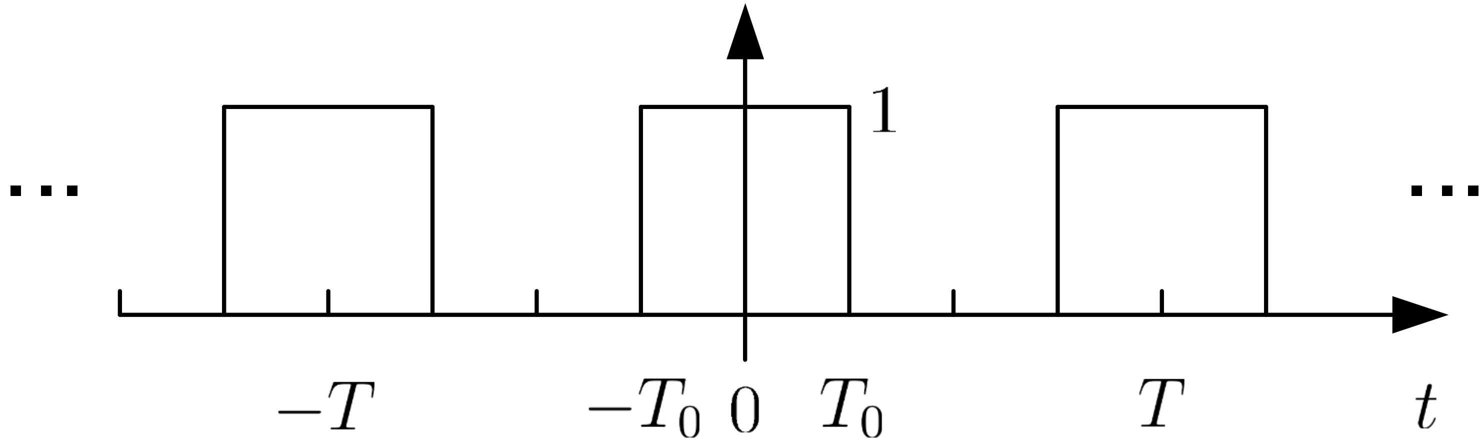


Fig.2.2: Periodic pulses

According to (2.3), the fundamental frequency is $\Omega_0 = 2\pi/T$. Using (2.5), we get:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T} \int_{-T_0}^{T_0} e^{-jk\Omega_0 t} dt$$

For $k = 0$:

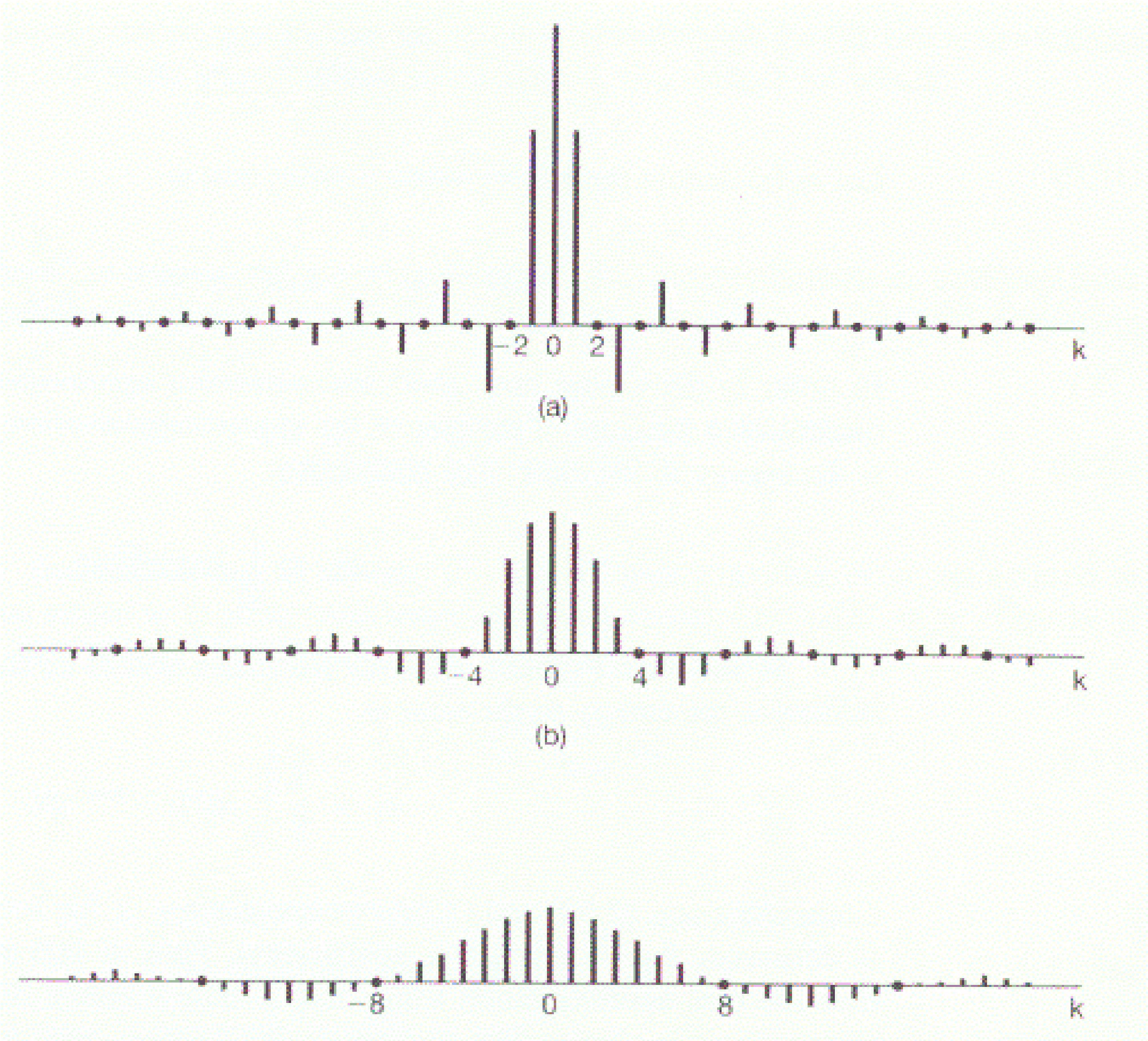
$$a_0 = \frac{1}{T}\int_{-T_0}^{T_0} 1 dt = \frac{2T_0}{T}$$

For $k \neq 0$:

$$a_k = \frac{1}{T}\int_{-T_0}^{T_0} e^{-jk\Omega_0 t} dt = -\frac{1}{jk\Omega_0 T} e^{-jk\Omega_0 t}\Big|_{-T_0}^{T_0} = \frac{\sin(k\Omega_0 T_0)}{k\pi} = \frac{\sin(2\pi k T_0/T)}{k\pi}$$

The reason of separating the cases of $k = 0$ and $k \neq 0$ is to facilitate the computation of a_0, whose value is not straightforwardly obtained from the general expression which involves "0/0". Nevertheless, using $\mathrm{L'H\hat{o}pital}$'s rule:

$$\lim_{k\to 0} \frac{\sin(2\pi k T_0/T)}{k\pi} = \lim_{k\to 0} \frac{\frac{d\sin(2\pi k T_0/T)}{dk}}{\frac{dk\pi}{dk}} = \lim_{k\to 0} \frac{2\pi T_0/T\cos((2\pi k T_0/T))}{\pi} = \frac{2T_0}{T}$$



$T = 4T_0$

$T = 8T_0$

$T = 16T_0$

In summary, if a signal $x(t)$ is continuous in time and periodic, we can write:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t}, \qquad t \in (-\infty, \infty) \tag{2.4}$$

The basic steps for finding the Fourier series coefficients are:

1. Determine the fundamental period T_p and fundamental frequency Ω_0

2. For all k, multiply $x(t)$ by $e^{-jk\Omega_0 t}$, then integrate with respect to t for one period, finally divide the result by T_p. Usually we separate the calculation into two cases: $k=0$ and $k \neq 0$

That is, $\{a_k\}$ correspond to the frequency domain representation of $x(t)$ and we may write:

$$x(t) \leftrightarrow X(j\Omega) \quad \text{or} \quad x(t) \leftrightarrow a_k \tag{2.1}$$

where $X(j\Omega)$, a function of frequency Ω, is characterized by $\{a_k\}$.

Both $x(t)$ and $X(j\Omega)$ represent the same signal: we observe the former in time domain while the latter in frequency domain.

How can you observe a time-domain signal in the frequency domain?

Fourier Transform

- For analysis of continuous-time aperiodic signals
- Defined on a continuous range of Ω

The Fourier transform of an aperiodic and continuous-time signal $x(t)$ is:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt \qquad (2.8)$$

which is also called spectrum. The inverse transform is given by

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \qquad (2.9)$$

Again, $x(t)$ and $X(j\Omega)$ represent the same signal

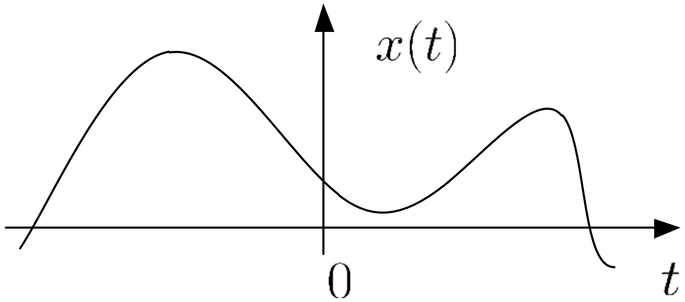
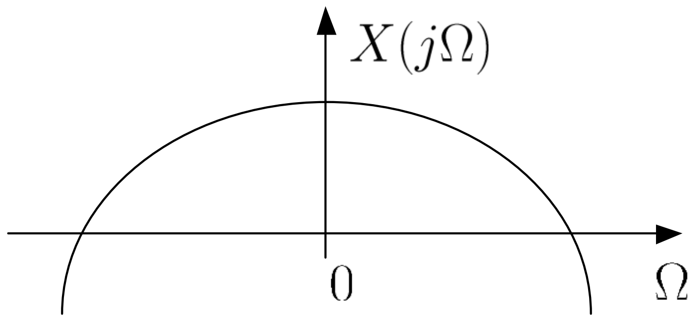
time domain	frequency domain
	
$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt$	
	$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega$
continuous and aperiodic	continuous and aperiodic

Fig.2.3: Illustration of Fourier transform

The delta function $\delta(t)$ has the following characteristics:

$$\delta(t) = 0, \quad t \neq 0 \tag{2.10}$$

$$\int_{-\infty}^{\infty} \delta(t) dt = 1 \tag{2.11}$$

and

$$f(t)\delta(t - t_0) = f(t_0)\delta(t - t_0) \tag{2.12}$$

where $f(t)$ is a continuous-time signal.

(2.10) and (2.11) indicate that $\delta(t)$ has a very large value or impulse at $t = 0$. That is, $\delta(t)$ is not well defined at $t = 0$

(2.12) is obtained by multiplying $f(t)$ by an impulse $\delta(t - t_0)$

$\delta(t)$ as the building block of any continuous-time signal, described by the sifting property:

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \quad (2.13)$$

That is, $x(t)$ can be considered as an "infinite sum" of impulse functions (multiplied by a zero width) at distinct times τ and each with amplitude $x(\tau)$

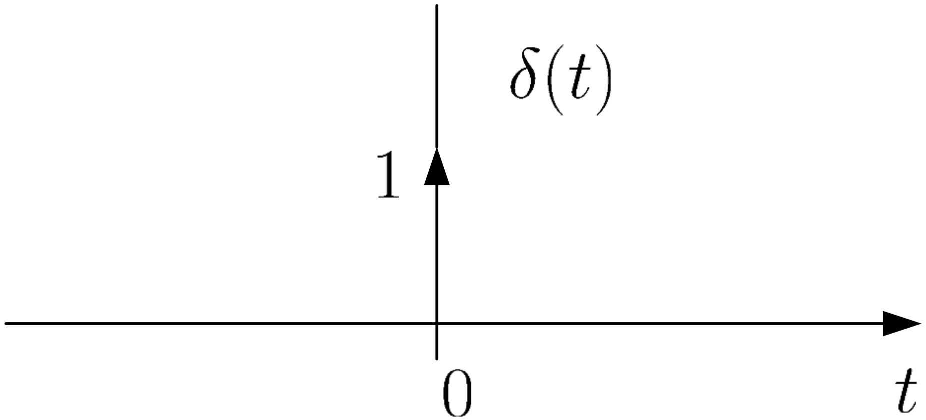


Fig.2.4: Representation of $\delta(t)$

The unit step function $u(t)$ has the form of:

$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases} \tag{2.14}$$

As there is a sudden change from 0 to 1 at $t = 0$, $u(0)$ is not well defined

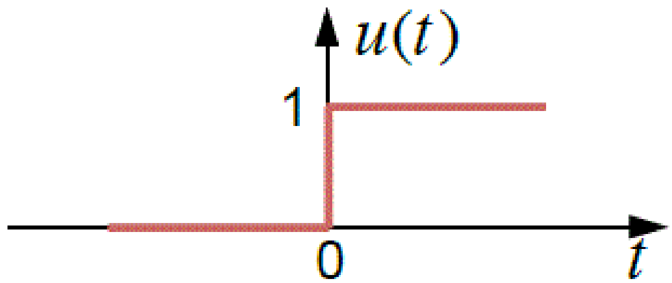


Fig. 2.5: Representation of $u(t)$

Example 2.4

Find the Fourier transform of $x(t)$ which is a rectangular pulse of the form:

$$x(t) = \begin{cases} 1, & -T_0 < t < T_0 \\ 0, & \text{otherwise} \end{cases}$$

Note that the signal is of finite length and corresponds to one period of the periodic function in Example 2.3. Applying (2.8) on $x(t)$ yields:

$$X(j\Omega) = \int_{-\infty}^{\infty} x(t) e^{-j\Omega t} dt = \int_{-T_0}^{T_0} e^{-j\Omega t} dt = \frac{2\sin(\Omega T_0)}{\Omega}$$

Define the sinc function as:

$$\mathrm{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

It is seen that $X(j\Omega)$ is a scaled sinc function because

$$X(j\Omega) = \frac{2\sin(\Omega T_0)}{\Omega} = 2T_0 \mathrm{sinc}\left(\frac{\Omega T_0}{\pi}\right)$$

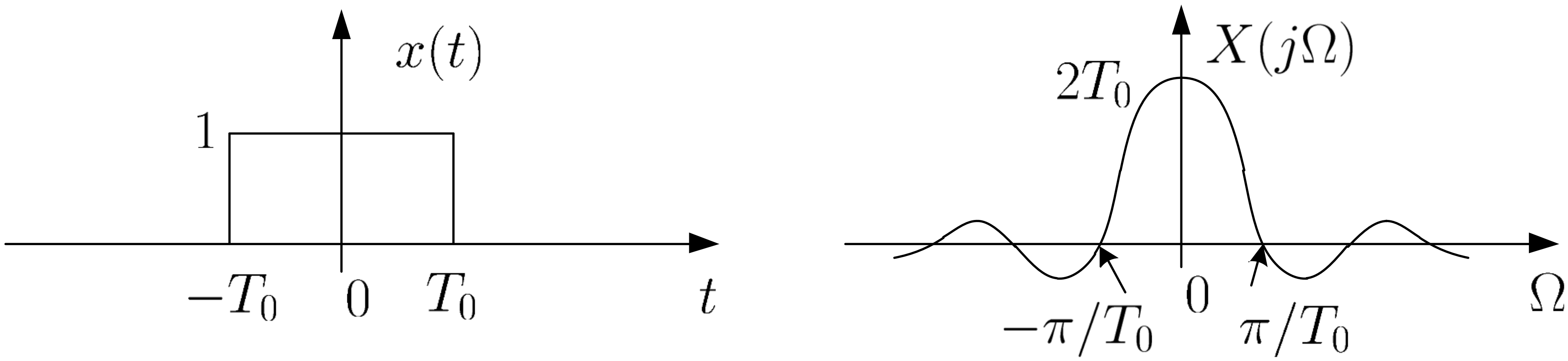


Fig.2.6: Fourier transform pair for rectangular pulse of $x(t)$

Example 2.5

Find the inverse Fourier transform of $X(j\Omega)$ which is a rectangular pulse of the form:

$$X(j\Omega) = \begin{cases} 1, & -W_0 < \Omega < W_0 \\ 0, & \text{otherwise} \end{cases}$$

Using (2.9), we get:

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega = \frac{1}{2\pi} \int_{-W_0}^{W_0} e^{j\Omega t} d\Omega = \frac{\sin(W_0 t)}{\pi t}$$

$$= \frac{W_0}{\pi} \text{sinc}\left(\frac{W_0 t}{\pi}\right)$$

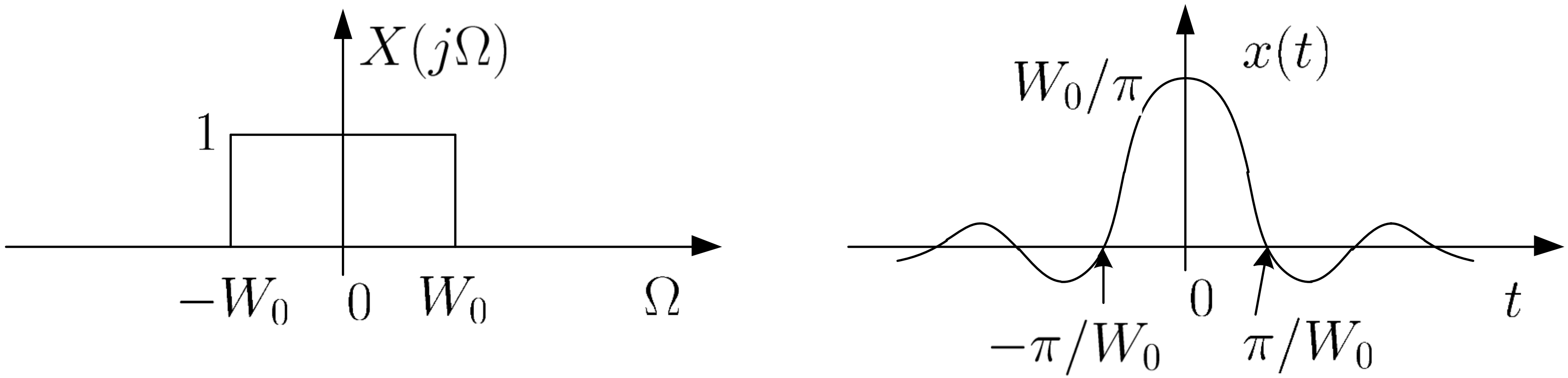


Fig.2.7: Fourier transform pair for rectangular pulse of $X(j\Omega)$

From Examples 2.4 and 2.5, we observe the duality property of Fourier transform

Can you guess why there is the duality property?

Example 2.6

Find the Fourier transform of $x(t) = e^{-at}u(t)$ with $a > 0$.

Employing the property of $u(t)$ in (2.14) and (2.8), we get:

$$X(j\Omega) = \int_0^\infty e^{-at} e^{-j\Omega t} dt = -\frac{1}{a+j\Omega} e^{-(a+j\Omega)t} \Big|_0^\infty = \frac{1}{a+j\Omega} = \frac{a-j\Omega}{a^2+\Omega^2}$$

Note that when $t \to \infty$, $e^{-at} \to 0$

$$|X(j\Omega)| = \frac{1}{\sqrt{a^2+\Omega^2}}$$

and

$$\angle(X(j\Omega)) = -\tan^{-1}\left(\frac{\Omega}{a}\right)$$

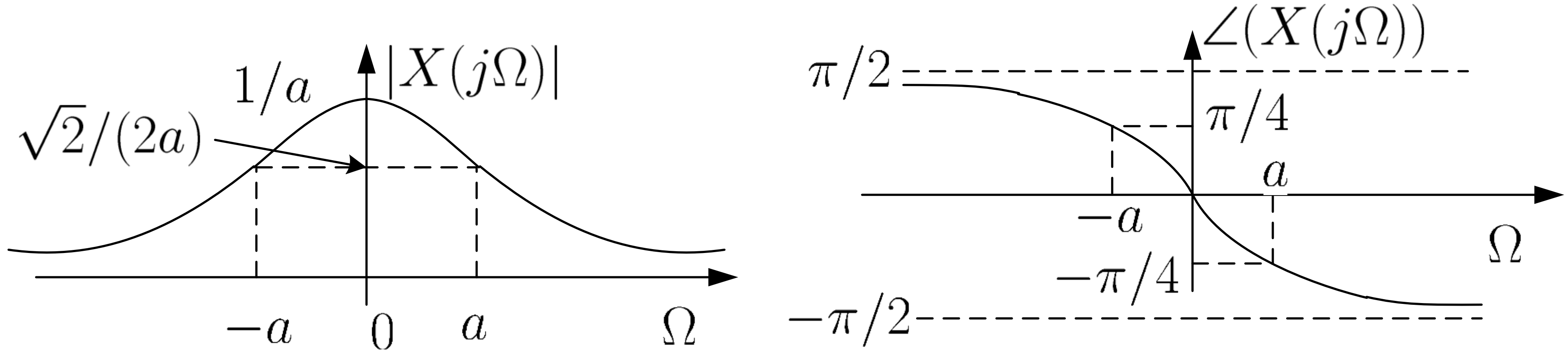


Fig.2.8: Magnitude and phase plots for $1/(a+j\Omega)$

Example 2.7

Find the Fourier transform of the delta function $x(t) = \delta(t)$.

Using (2.11) and (2.12) with $f(t) = e^{-j\Omega t}$ and $t_0 = 0$, we get:

$$X(j\Omega) = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega t} dt = \int_{-\infty}^{\infty} \delta(t) e^{-j\Omega \cdot 0} dt = e^{-j\Omega \cdot 0} \int_{-\infty}^{\infty} \delta(t) dt = e^{-j\Omega \cdot 0} = 1$$

Spectrum of $\delta(t)$ has unit amplitude at all frequencies

Based on $\delta(t)$, Fourier transform can be used to represent continuous-time periodic signals. Consider

$$X(j\Omega) = 2\pi\delta(\Omega - \Omega_0) \qquad (2.15)$$

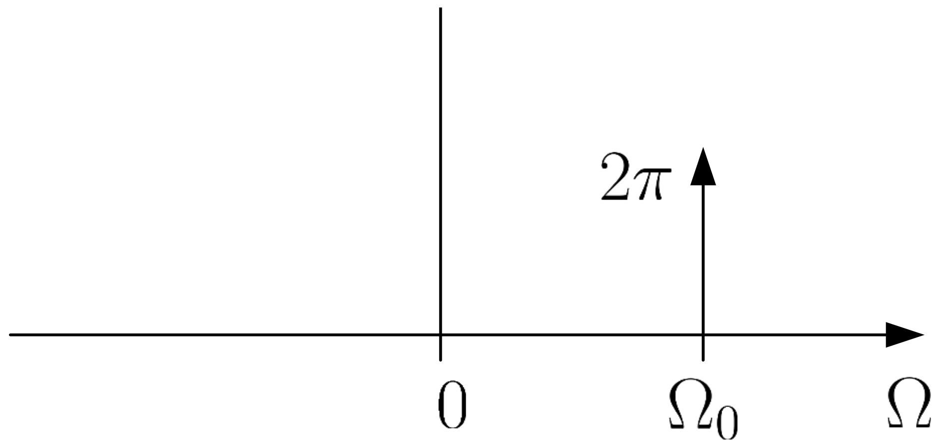


Fig.2.9: Impulse in frequency domain

Taking the inverse Fourier transform of $X(j\Omega)$ and employing Example 2.7, $x(t)$ is computed as:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi\delta(\Omega - \Omega_0)e^{j\Omega t}d\Omega = e^{j\Omega_0 t} \qquad (2.16)$$

As a result, the Fourier transform pair is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0) \qquad (2.17)$$

From (2.4) and (2.17), the Fourier transform pair for a continuous-time periodic signal is:

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} \leftrightarrow \sum_{k=-\infty}^{\infty} 2\pi a_k \delta(\Omega - k\Omega_0) \qquad (2.18)$$

<u>Example 2.8</u>

Find the Fourier transform of $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$ which is called an impulse train.

Clearly, $x(t)$ is a periodic signal with a period of T. Using (2.5) and Example 2.7, the Fourier series coefficients are:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \delta(t) e^{-jk\Omega_0 t} dt = \frac{1}{T}$$

with $\Omega_0 = 2\pi/T$. According to (2.18), the Fourier transform is:

$$X(j\Omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta\left(\Omega - \frac{2\pi k}{T}\right) = \Omega_0 \sum_{k=-\infty}^{\infty} \delta(\Omega - k\Omega_0)$$

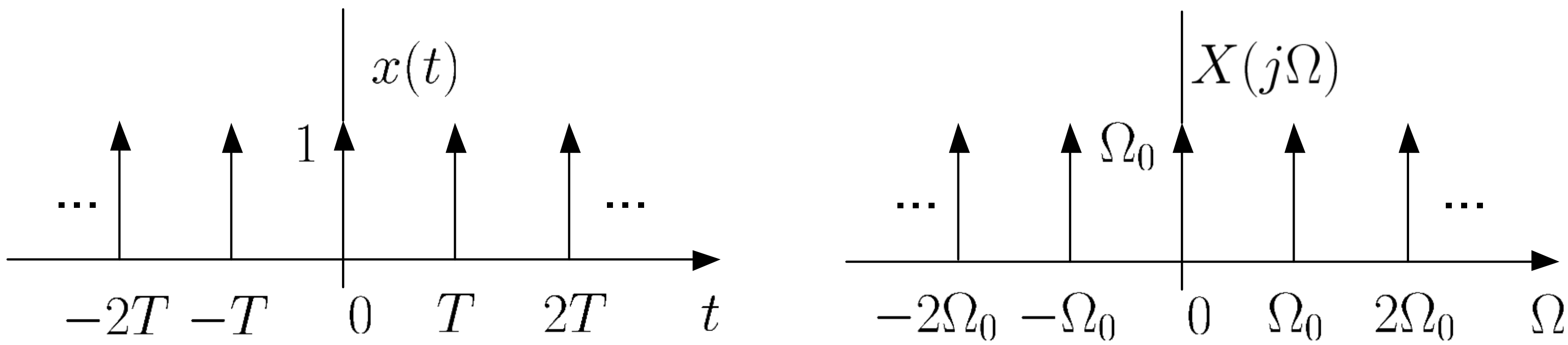


Fig.2.10: Fourier transform pair for impulse train

Fourier transform can be derived from Fourier series:

Consider $x(t)$ and $\tilde{x}(t)$:

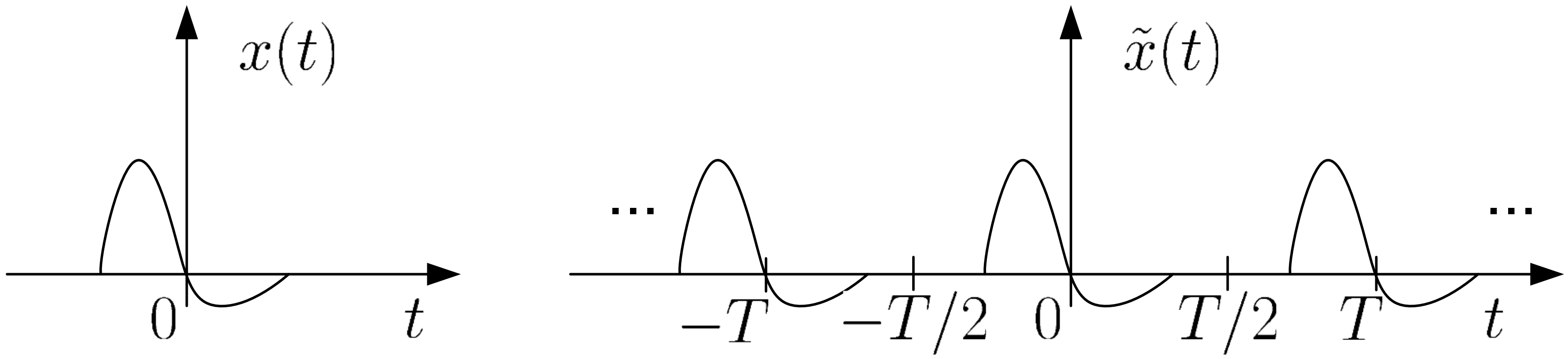


Fig.2.11: Constructing $\tilde{x}(t)$ from $x(t)$

$\tilde{x}(t)$ is constructed as a periodic version of $x(t)$, with period T

According to (2.5), the Fourier series coefficients of $\tilde{x}(t)$ are:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} \tilde{x}(t) e^{-jk\Omega_0 t} dt \tag{2.19}$$

where $\Omega_0 = 2\pi/T$. Noting that $x(t) = \tilde{x}(t)$ for $|t| < T/2$ and $x(t) = 0$ for $|t| > T/2$, (2.18) can be expressed as:

$$a_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t) e^{-jk\Omega_0 t} dt = \frac{1}{T} \int_{-\infty}^{\infty} x(t) e^{-jk\Omega_0 t} dt \tag{2.20}$$

According to (2.8), we can express a_k as:

$$a_k = \frac{1}{T} X(jk\Omega_0) \quad (2.21)$$

The Fourier series expansion for $\tilde{x}(t)$ is thus:

$$\tilde{x}(t) = \sum_{k=-\infty}^{\infty} \frac{1}{T} X(jk\Omega_0) e^{jk\Omega_0 t} = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t} \quad (2.22)$$

Considering $\tilde{x}(t) \to x(t)$ as $T \to \infty$ or $\Omega_0 \to 0$ and $\Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t}$ as the area of a rectangle whose height is $X(jk\Omega_0) e^{jk\Omega_0 t}$ and width corresponds to the interval of $[k\Omega_0, (k+1)\Omega_0]$, we obtain

$$x(t) = \lim_{\Omega_0 \to 0} \tilde{x}(t) = \lim_{\Omega_0 \to 0} \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \Omega_0 X(jk\Omega_0) e^{jk\Omega_0 t} = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(j\Omega) e^{j\Omega t} d\Omega \quad (2.23)$$

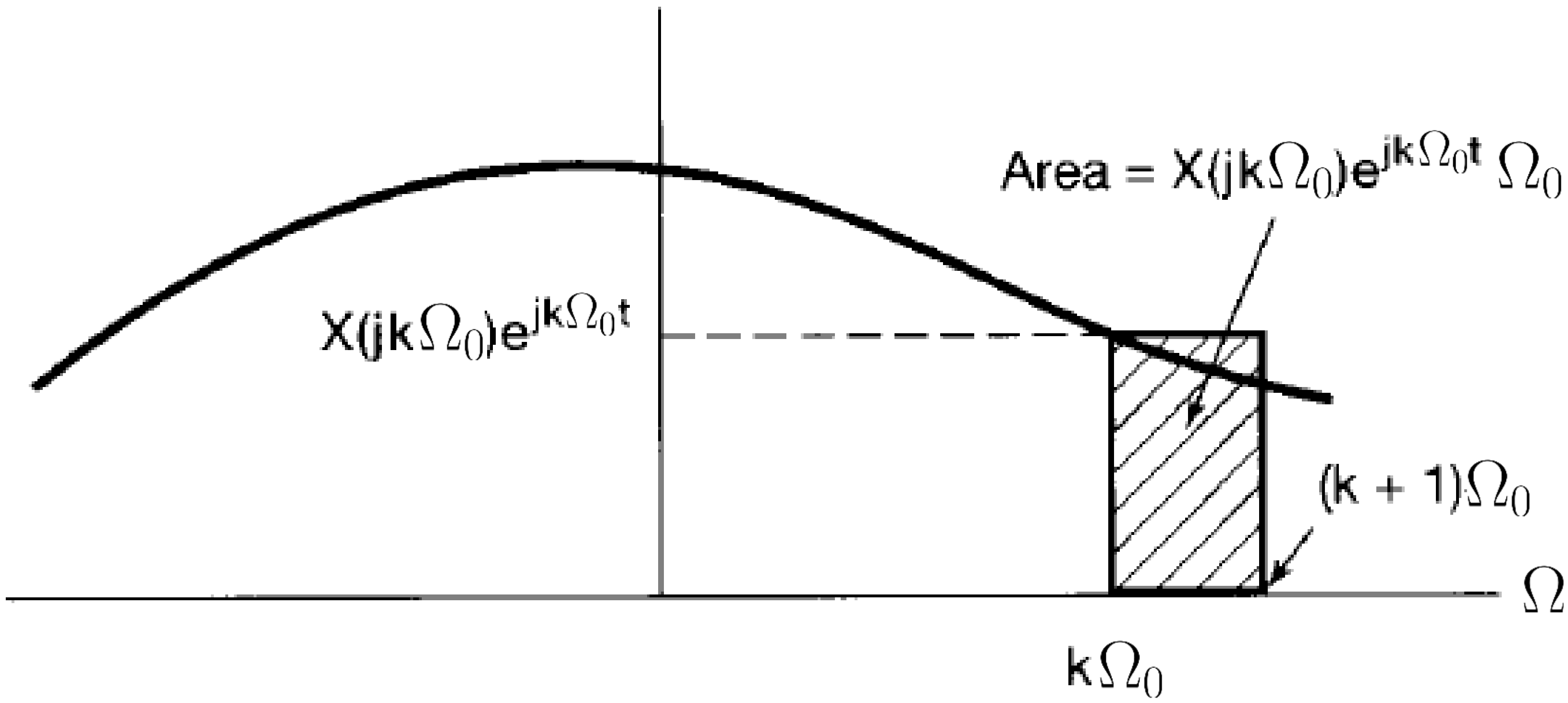


Fig. 2.12: Fourier transform from Fourier series

Linear Time-Invariant (LTI) System

- Linearity: if $(x_1(t), y_1(t))$ and $(x_2(t), y_2(t))$ are two continuous-time input-output pairs, then $ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$
- Time-Invariance: if $x(t) \rightarrow y(t)$, then $x(t - t_0) \rightarrow y(t - t_0)$
- Impulse response $h(t)$ is continuous-time signal which is the output of a continuous-time LTI system when the input is the impulse $\delta(t)$, and it can indicate the system functionality

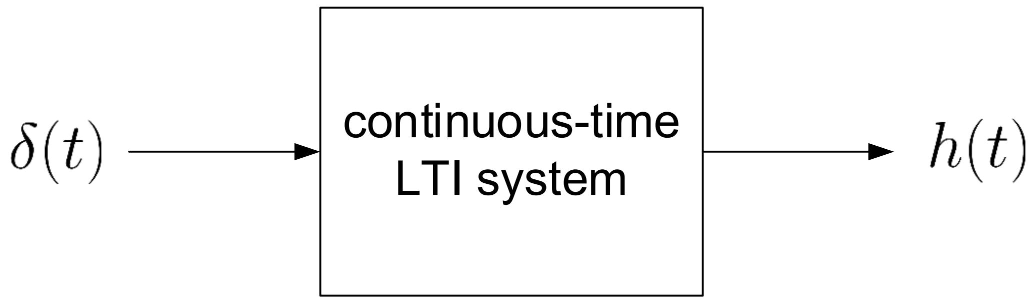


Fig. 2.13: Continuous-time impulse response

- The input-output relationship for a LTI system is characterized by convolution:

$$y(t) = x(t) \otimes h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \quad (2.24)$$

where $x(t)$, $y(t)$ and $h(t)$ are input, output and impulse response, respectively

Example 2.9

Determine the function of a LTI continuous-time system if its impulse response is $h(t) = 0.1[u(t) - u(t-10)]$.

Using (2.24), we get:

$$
\begin{aligned}
y(t) &= h(t) \otimes x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \\
&= 0.1 \int_{-\infty}^{\infty} [u(\tau) - u(\tau - 10)]x(t-\tau)d\tau \\
&= \frac{1}{10} \int_{0}^{10} x(t-\tau)d\tau
\end{aligned}
$$

Note that $[u(\tau) - u(\tau - 10)]$ is a rectangular pulse for $\tau \in (0, 10)$.

The system computes average input value from the current time minus 10 to current time.

- Convolution in time domain corresponds to multiplication in Fourier transform domain, i.e.,

$$
x(t) \otimes h(t) \leftrightarrow X(j\Omega)H(j\Omega) \tag{2.25}
$$

Proof:

The Fourier transform of $x(t) \otimes h(t)$ is

$$\begin{aligned} Y(j\Omega) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau) h(t-\tau) e^{-j\Omega t} d\tau dt \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(t-\tau) e^{-j\Omega t} dt \right] d\tau \\ &= \int_{-\infty}^{\infty} x(\tau) \left[\int_{-\infty}^{\infty} h(u) e^{-j\Omega(u+\tau)} du \right] d\tau, \quad u = t - \tau \\ &= \left[\int_{-\infty}^{\infty} x(\tau) e^{-j\Omega\tau} d\tau \right] \cdot \left[\int_{-\infty}^{\infty} h(u) e^{-j\Omega u} du \right] \\ &= X(j\Omega) \cdot H(j\Omega) \end{aligned} \tag{2.26}$$

This suggests that $y(t)$ can be computed from inverse Fourier transform of $X(j\Omega)H(j\Omega)$.