z Transform

Chapter Intended Learning Outcomes:

(i) Understanding the relationship between z transform and the Fourier transform for discrete-time signals

(ii) Understanding the characteristics and properties of z transform

(iii) Ability to compute z transform and inverse z transform

(iv) Ability to apply z transform for analyzing linear time-invariant (LTI) systems

Definition

The z transform of $x[n]$, denoted by $X(z)$, is defined as:

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] z^{-n} \qquad (5.1)$$

where z is a continuous complex variable.

Relationship with Fourier Transform

Employing (4.2), we construct the continuous-time sampled signal $x_s(t)$ with a sampling interval of T from $x[n]$:

$$x_s(t) = \sum_{n=-\infty}^{\infty} x[n] \delta(t - nT) \qquad (5.2)$$

Taking Fourier transform of $x_s(t)$ with using properties of $\delta(t)$:

$$X_s(j\Omega) = \int_{-\infty}^{\infty} x_s(t)e^{-j\Omega t}dt = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} x[n]\delta(t-nT)e^{-j\Omega t}dt$$

$$= \sum_{n=-\infty}^{\infty} x[n]e^{-j\Omega nT} \quad (5.3)$$

Defining $\omega = \Omega T$ as the discrete-time frequency parameter and writing $X_s(j\Omega)$ as $X(e^{j\omega})$, (5.3) becomes

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \quad (5.4)$$

which is known as discrete-time Fourier transform (DTFT) or Fourier transform of discrete-time signals

$X(e^{j\omega})$ is periodic with period 2π:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} = \sum_{n=-\infty}^{\infty} x[n]e^{-j(\omega+2k\pi)n} = X(e^{j(\omega+2k\pi)}) \quad (5.5)$$

where k is any integer. Since z is a continuous complex variable, we can write

$$z = re^{j\omega} \quad (5.6)$$

where $r = |z| > 0$ is magnitude and $\omega = \angle(z)$ is angle of z. Employing (5.6), the z transform is:

$$X(z)|_{z=re^{j\omega}} = X(re^{j\omega}) = \sum_{n=-\infty}^{\infty} \left(x[n]r^{-n}\right) e^{-j\omega n} \quad (5.7)$$

which is equal to the DTFT of $x[n]r^{-n}$. When $r = 1$ or $z = e^{j\omega}$, (5.7) and (5.4) are identical:

$$X(z)\big|_{z=e^{j\omega}} = X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \tag{5.8}$$

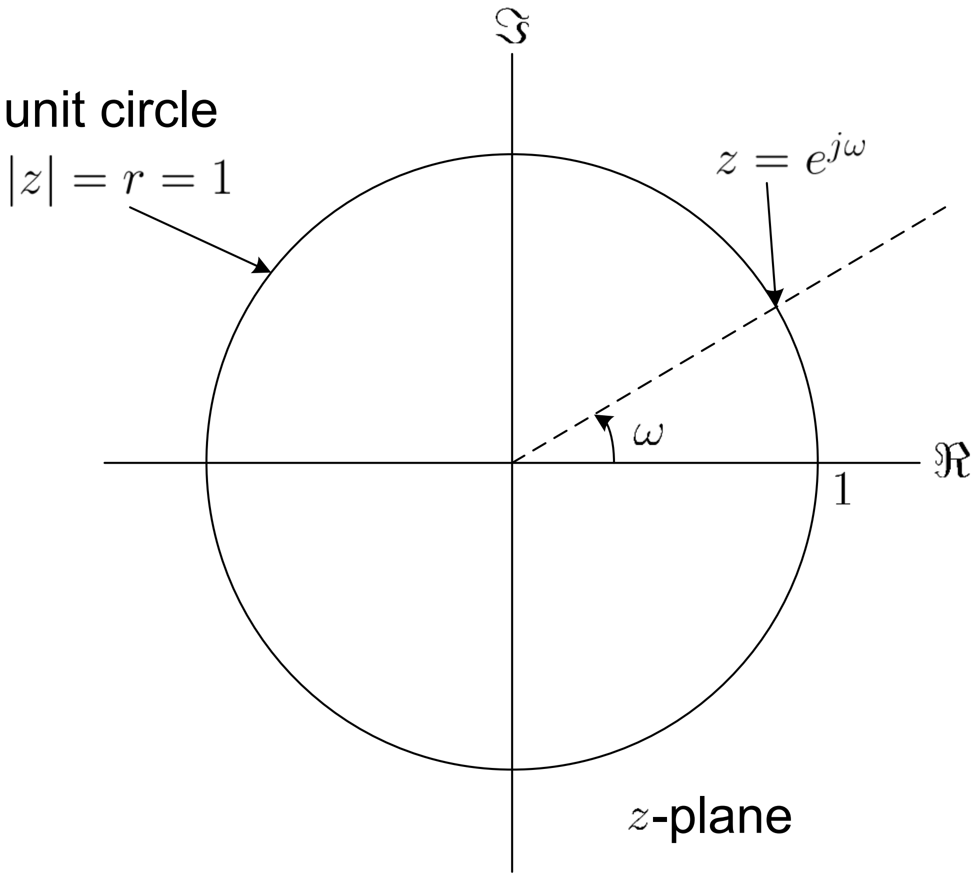


Fig.5.1: Relationship between $X(z)$ and $X(e^{j\omega})$ on the z-plane

Region of Convergence (ROC)

ROC indicates when z transform of a sequence converges

Generally there exists some z such that

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \rightarrow \infty \qquad (5.9)$$

where the z transform does not converge

The set of values of z for which $X(z)$ converges or

$$|X(z)| = \left| \sum_{n=-\infty}^{\infty} x[n]z^{-n} \right| \leq \sum_{n=-\infty}^{\infty} \left| x[n]z^{-n} \right| < \infty \qquad (5.10)$$

is called the ROC, which must be specified along with $X(z)$ in order for the z transform to be complete

Assuming that $x[n]$ is of infinite length, we decompose $X(z)$:

$$X(z) = X_-(z) + X_+(z) \tag{5.11}$$

where

$$X_-(z) = \sum_{n=-\infty}^{-1} x[n]z^{-n} = \sum_{m=1}^{\infty} x[-m]z^{m} \tag{5.12}$$

and

$$X_+(z) = \sum_{n=0}^{\infty} x[n]z^{-n} \tag{5.13}$$

Let $f_n(z) = x[n]z^{-n}$, $X_+(z)$ is expanded as:

$$\begin{aligned} X_+(z) &= x[0]z^{-0} + x[1]z^{-1} + \cdots + x[n]z^{-n} + \cdots \\ &= f_0(z) + f_1(z) + \cdots + f_n(z) + \cdots \end{aligned} \tag{5.14}$$

According to the ratio test, convergence of $X_+(z)$ requires

$$\lim_{n\to\infty}\left|\frac{f_{n+1}(z)}{f_n(z)}\right| < 1 \tag{5.15}$$

Let $\lim_{n\to\infty}|x[n+1]/x[n]| = R_+ > 0$. $X_+(z)$ converges if

$$\begin{aligned}&\lim_{n\to\infty}\left|\frac{x[n+1]z^{-n-1}}{x[n]z^{-n}}\right| = \lim_{n\to\infty}\left|\frac{x[n+1]}{x[n]}\right||z^{-1}| < 1\\ &\Rightarrow |z| > \lim_{n\to\infty}\left|\frac{x[n+1]}{x[n]}\right| = R_+\end{aligned} \tag{5.16}$$

That is, the ROC for $X_+(z)$ is $|z| > R_+$.

Let $\lim_{m\to\infty} |x[-m]/x[-m-1]| = R_- > 0$. $X_-(z)$ converges if

$$\lim_{m\to\infty} \left| \frac{x[-m-1]z^{m+1}}{x[-m]z^m} \right| = \lim_{m\to\infty} \left| \frac{x[-m-1]}{x[-m]} \right| |z| < 1$$

$$\Rightarrow |z| < \lim_{m\to\infty} \left| \frac{x[-m]}{x[-m-1]} \right| = R_- \quad (5.17)$$

As a result, the ROC for $X_-(z)$ is $|z| < R_-$

Combining the results, the ROC for $X(z)$ is $R_+ < |z| < R_-$:

- ROC is a ring when $R_+ < R_-$
- No ROC if $R_- < R_+$ and $X(z)$ does not exist

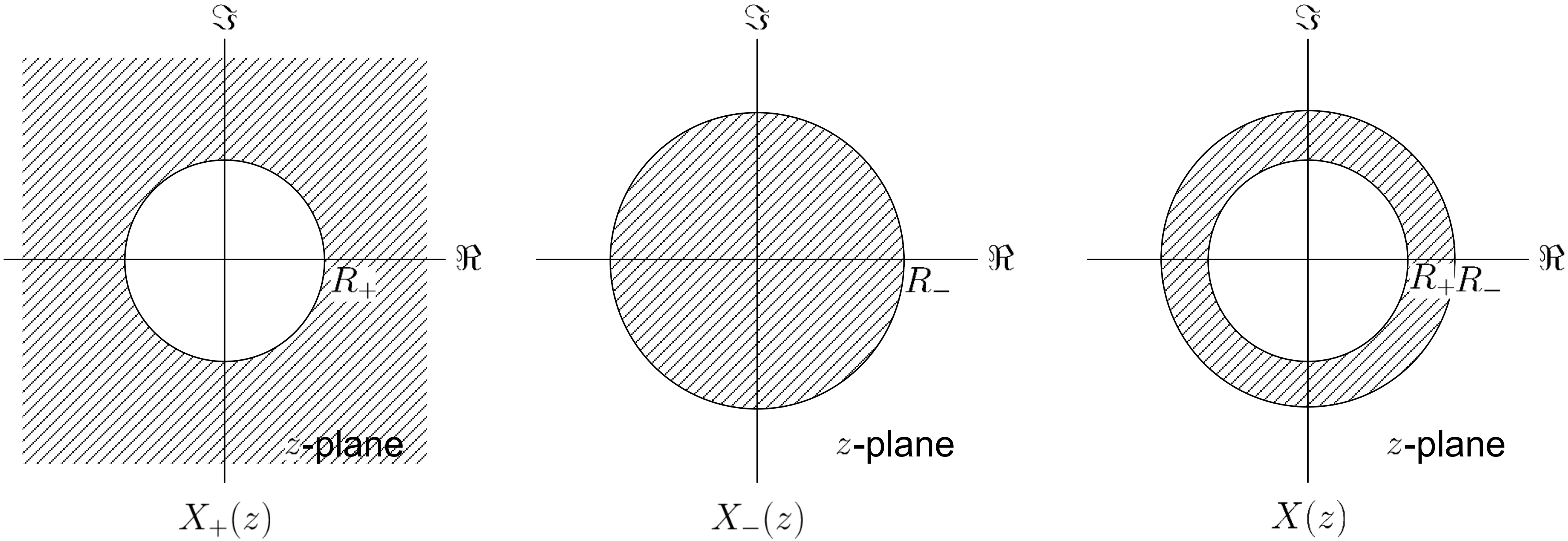


Fig.5.2: ROCs for $X_+(z)$, $X_-(z)$ and $X(z)$

Poles and Zeros

Values of z for which $X(z) = 0$ are the zeros of $X(z)$

Values of z for which $X(z) = \pm\infty$ are the poles of $X(z)$

In many real-world applications, $X(z)$ is represented as a rational function:

$$X(z) = \frac{P(z)}{Q(z)} = \frac{\sum_{k=0}^{M} b_k z^k}{\sum_{k=0}^{N} a_k z^k} \tag{5.18}$$

Factorizing $P(z)$ and $Q(z)$, (5.18) can be written as

$$X(z) = \frac{b_0(z-d_1)(z-d_2)\cdots(z-d_M)}{a_0(z-c_1)(z-c_2)\cdots(z-c_N)} \tag{5.19}$$

How many poles and zeros in (5.18)? What are they?

Example 5.1

Determine the z transform of $x[n]=a^n u[n]$ where $u[n]$ is the unit step function. Then determine the condition when the DTFT of $x[n]$ exists.

Using (5.1) and (3.3), we have

$$X(z)=\sum_{n=-\infty}^{\infty} a^n u[n] z^{-n}=\sum_{n=0}^{\infty}\left(az^{-1}\right)^n$$

According to (5.10), $X(z)$ converges if

$$\sum_{n=0}^{\infty}\left|az^{-1}\right|^n<\infty$$

Applying the ratio test, the convergence condition is

$$\left|az^{-1}\right|<1 \Leftrightarrow |z|>|a|$$

Note that we cannot write $|z| > a$ because a may be complex

For $|z| > |a|$, $X(z)$ is computed as

$$X(z) = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1 - \left(az^{-1}\right)^{\infty}}{1 - az^{-1}} = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

Together with the ROC, the z transform of $x[n] = a^n u[n]$ is:

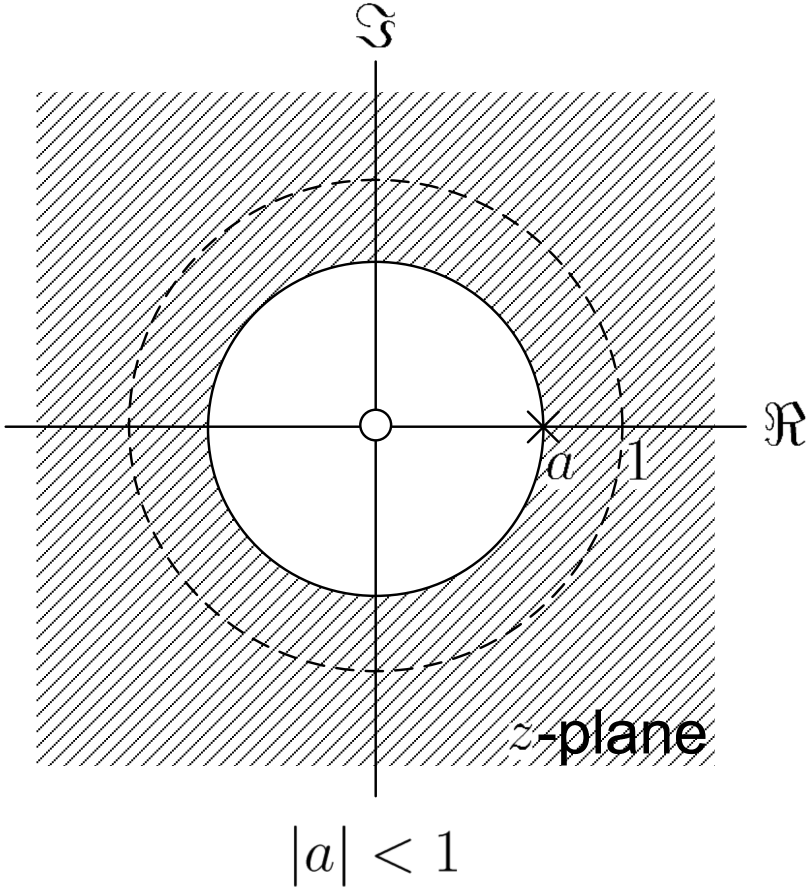
$$X(z) = \frac{z}{z - a}, \quad |z| > |a|$$

It is clear that $X(z)$ has a zero at $z = 0$ and a pole at $z = a$. Using (5.8), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 > |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| < 1$.

Otherwise, its DTFT does not exist. In general, the DTFT $X(e^{j\omega})$ exists if its ROC includes the unit circle. If $|z|>|a|$ includes $|z|=1$, $|a|<1$ is required.



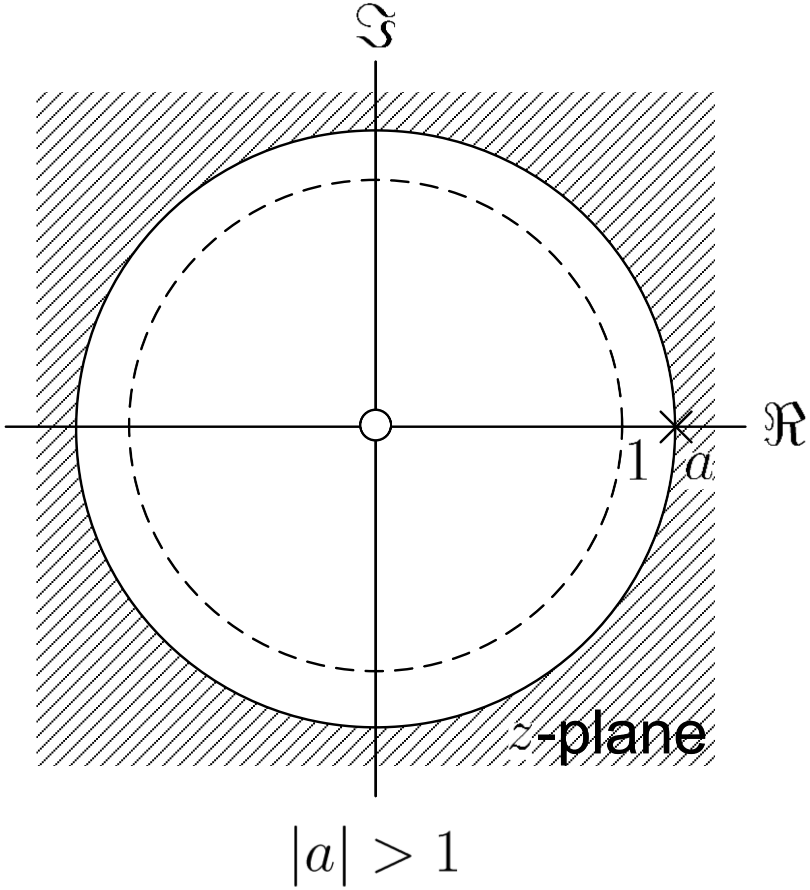


Fig.5.3: ROCs for $|a|<1$ and $|a|>1$ when $x[n]=a^n u[n]$

Example 5.2

Determine the z transform of $x[n] = -a^n u[-n-1]$. Then determine the condition when the DTFT of $x[n]$ exists.

Using (5.1) and (3.3), we have

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{m=1}^{\infty} a^{-m} z^m = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m$$

Similar to Example 5.1, $X(z)$ converges if $\left|a^{-1}z\right| < 1$ or $|z| < |a|$, which aligns with the ROC for $X_{-}(z)$ in (5.17). This gives

$$X(z) = -\sum_{m=1}^{\infty} \left(a^{-1}z\right)^m = -\frac{a^{-1}z\left(1-\left(a^{-1}z\right)^{\infty}\right)}{1-a^{-1}z} = -\frac{a^{-1}z}{1-a^{-1}z} = \frac{z}{z-a}$$

Together with ROC, the z transform of $x[n] = -a^n u[-n-1]$ is:

$$X(z) = \frac{z}{z-a}, \quad |z| < |a|$$

Using (5.8), we substitute $z = e^{j\omega}$ to obtain

$$X(e^{j\omega}) = \frac{e^{j\omega}}{e^{j\omega} - a}, \quad |e^{j\omega}| = 1 < |a|$$

As a result, the existence condition for DTFT of $x[n]$ is $|a| > 1$.

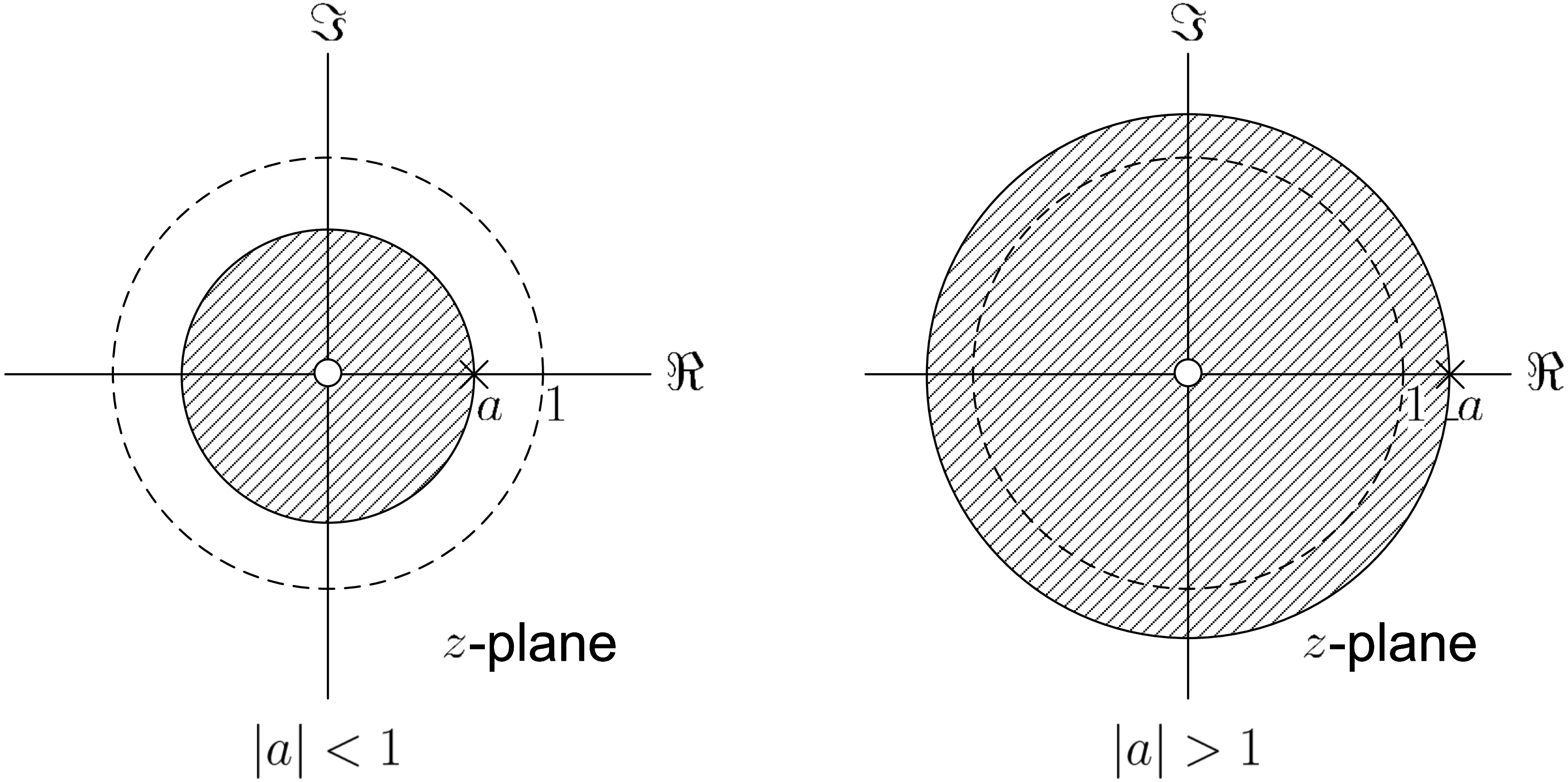


Fig.5.4: ROCs for $|a| < 1$ and $|a| > 1$ when $x[n] = -a^n u[-n-1]$

Example 5.3

Determine the z transform of $x[n] = a^n u[n] + b^n u[-n-1]$ where $|a| < |b|$.

Employing the results in Examples 5.1 and 5.2, we have

$$\begin{aligned} X(z) &= \frac{1}{1-az^{-1}} + \left(-\frac{1}{1-bz^{-1}}\right), \quad |z| > |a| \quad \text{and} \quad |z| < |b| \\ &= \frac{(a-b)z^{-1}}{(1-az^{-1})(1-bz^{-1})} \\ &= \frac{(a-b)z}{(z-a)(z-b)}, \quad |a| < |z| < |b| \end{aligned}$$

Note that its ROC agrees with Fig.5.2.

What are the pole(s) and zero(s) of *X*(*z*)?

Example 5.4

Determine the z transform of $x[n] = \delta[n+1]$.

Using (5.1) and (3.2), we have

$$X(z) = \sum_{n=-\infty}^{\infty} \delta[n+1] z^{-n} = z$$

Example 5.5

Determine the z transform of $x[n]$ which has the form of:

$$x[n] = \begin{cases} a^n, & 0 \le n \le N-1 \\ 0, & \text{otherwise} \end{cases}$$

Using (5.1), we have

$$X(z) = \sum_{n=0}^{N-1} \left(az^{-1}\right)^n = \frac{1-\left(az^{-1}\right)^N}{1-az^{-1}} = \frac{1}{z^{N-1}} \frac{z^N - a^N}{z-a}$$

What are the ROCs in Examples 5.4 and 5.5?

Finite-Duration and Infinite-Duration Sequences

Finite-duration sequence: values of $x[n]$ are nonzero only for a finite time interval

Otherwise, $x[n]$ is called an infinite-duration sequence:

- Right-sided: if $x[n]=0$ for $n<N_+<\infty$ where N_+ is an integer (e.g., $x[n]=a^n u[n]$ with $N_+=0$; $x[n]=a^n u[n-10]$ with $N_+=10$; $x[n]=a^n u[n+10]$ with $N_+=-10$)
- Left-sided: if $x[n]=0$ for $n>N_->-\infty$ where N_- is an integer (e.g., $x[n]=-a^n u[-n-1]$ with $N_-=-1$)
- Two-sided: neither right-sided nor left-sided (e.g., Example 5.3)

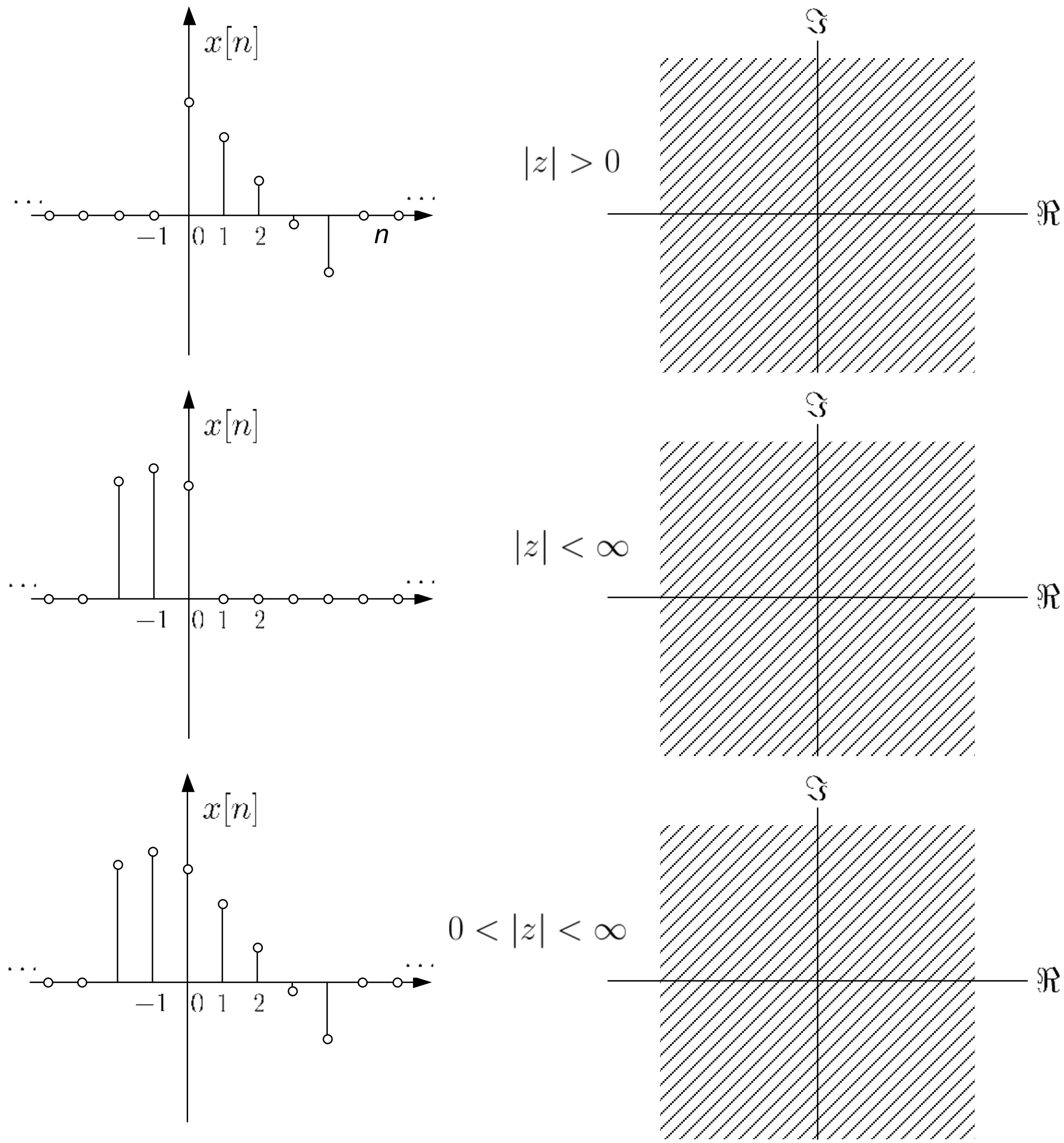


Fig.5.5: Finite-duration sequences

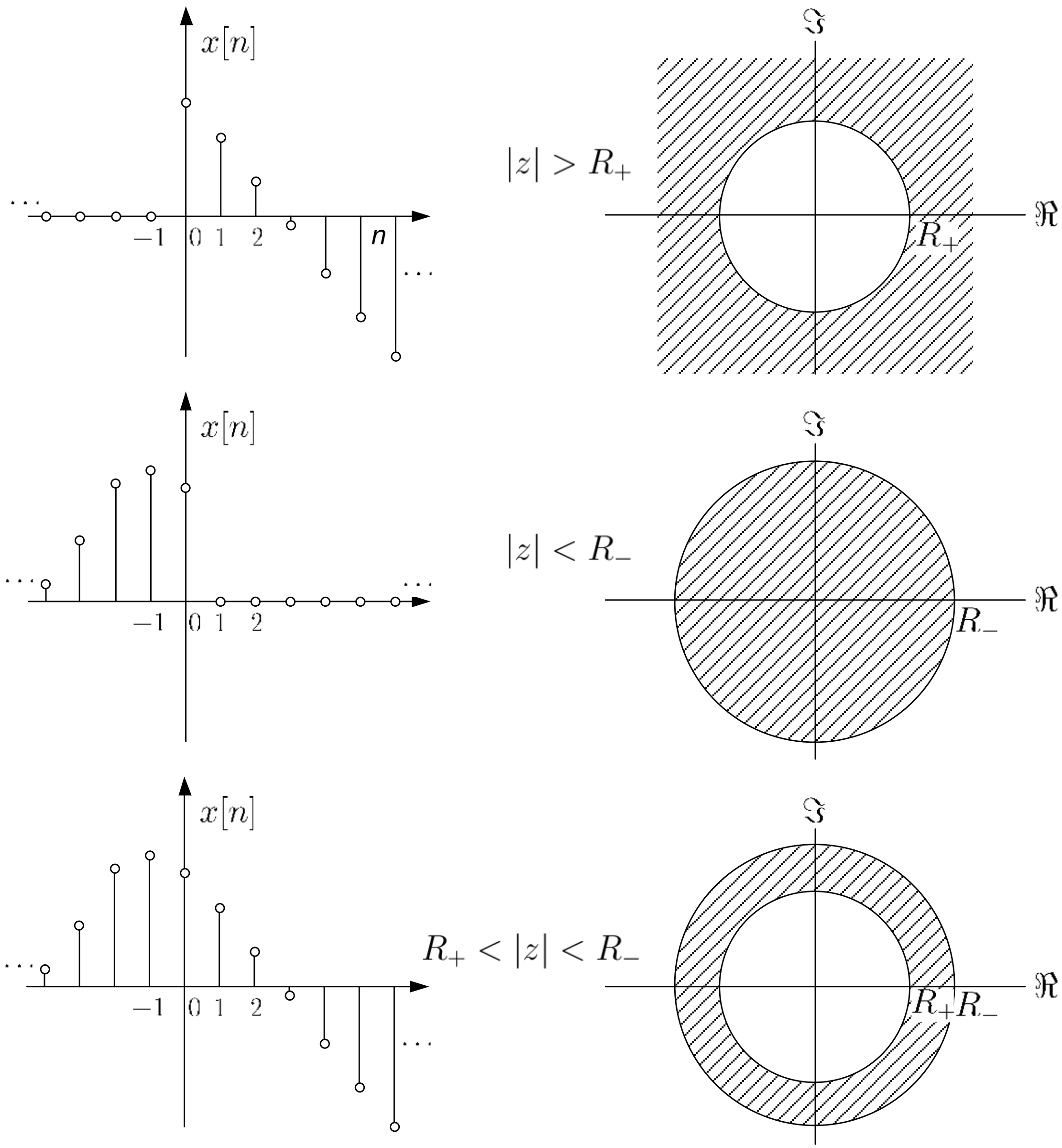


Fig. 5.6: Infinite-duration sequences

Sequence	Transform	ROC
$\delta[n]$	1	All z
$\delta[n-m]$	z^{-m}	$\lvert z\rvert > 0$, $m>0$; $\lvert z\rvert < \infty$, $m<0$
$a^n u[n]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert > \lvert a\rvert$
$-a^n u[-n-1]$	$\dfrac{1}{1-az^{-1}}$	$\lvert z\rvert < \lvert a\rvert$
$na^n u[n]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert > \lvert a\rvert$
$-na^n u[-n-1]$	$\dfrac{az^{-1}}{(1-az^{-1})^2}$	$\lvert z\rvert < \lvert a\rvert$
$a^n \cos(bn) u[n]$	$\dfrac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$
$a^n \sin(bn) u[n]$	$\dfrac{a\sin(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}$	$\lvert z\rvert > \lvert a\rvert$

Table 5.1: z transforms for common sequences

Eight ROC properties are:

P1. There are four possible shapes for ROC, namely, the entire region except possibly $z = 0$ and/or $z = \infty$, a ring, or inside or outside a circle in the z-plane centered at the origin (e.g., Figures 5.5 and 5.6)

P2. The DTFT of a sequence $x[n]$ exists if and only if the ROC of the z transform of $x[n]$ includes the unit circle (e.g., Examples 5.1 and 5.2)

P3: The ROC cannot contain any poles (e.g., Examples 5.1 to 5.5)

P4: When $x[n]$ is a finite-duration sequence, the ROC is the entire z-plane except possibly $z = 0$ and/or $z = \pm\infty$ (e.g., Examples 5.4 and 5.5)

P5: When $x[n]$ is a right-sided sequence, the ROC is of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the pole with the largest magnitude in $X(z)$ (e.g., Example 5.1)

P6: When $x[n]$ is a left-sided sequence, the ROC is of the form $|z| < |p_{\min}|$ where $p_{\min}$ is the pole with the smallest magnitude in $X(z)$ (e.g., Example 5.2)

P7: When $x[n]$ is a two-sided sequence, the ROC is of the form $|p_a| < |z| < |p_b|$ where p_a and p_b are two poles with the successive magnitudes in $X(z)$ such that $|p_a| < |p_b|$ (e.g., Example 5.3)

P8: The ROC must be a connected region

<u>Example 5.6</u>

A z transform $X(z)$ contains three poles, namely, a, b and c with $|a| < |b| < |c|$. Determine all possible ROCs.

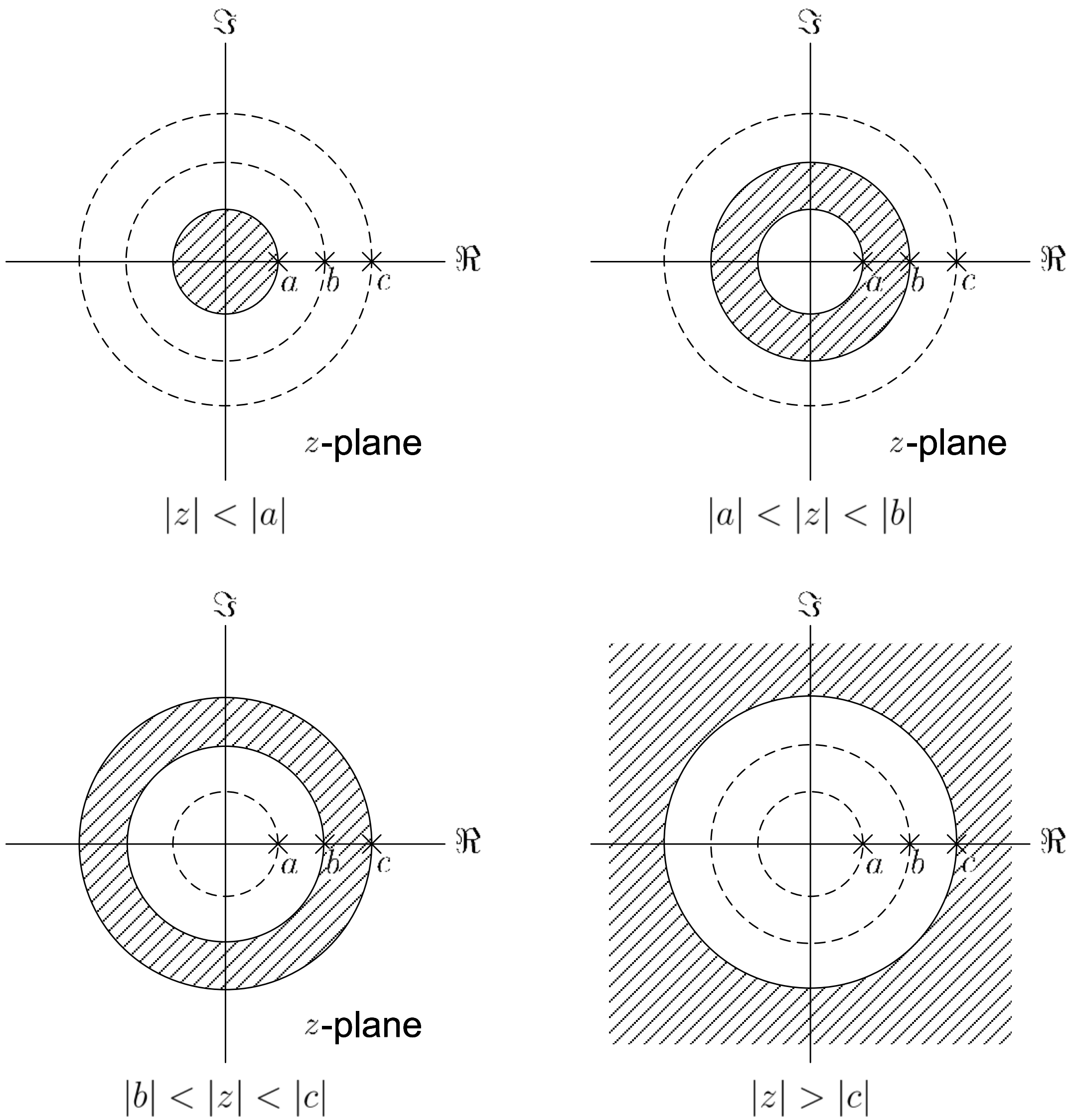


Fig.5.7: ROC possibilities for three poles

What are other possible ROCs?

Properties of z Transform

1. Linearity

Let $(x_1[n], X_1(z))$ and $(x_2[n], X_2(z))$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively, we have

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z) \tag{5.20}$$

Its ROC is denoted by $\mathcal{R}$, which includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$ where $\cap$ is the intersection operator. That is, $\mathcal{R}$ contains at least the intersection of $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$.

Example 5.7

Determine the z transform of $y[n]$ which is expressed as:

$$y[n] = x_1[n] + x_2[n]$$

where $x_1[n] = (0.2)^n u[n]$ and $x_2[n] = (-0.3)^n u[n]$.

From Table 5.1, the z transforms of $x_1[n]$ and $x_2[n]$ are:

$$x_1[n] = (0.2)^n u[n] \leftrightarrow \frac{1}{1 - 0.2z^{-1}}, \quad |z| > 0.2$$

and

$$x_2[n] = (-0.3)^n u[n] \leftrightarrow \frac{1}{1 + 0.3z^{-1}}, \quad |z| > 0.3$$

According to the linearity property, the z transform of $y[n]$ is

$$Y(z) = \frac{1}{1 - 0.2z^{-1}} + \frac{1}{1 + 0.3z^{-1}}, \quad |z| > 0.3$$

Why the ROC is |z|>0.3 instead of |z|>0.2?

2. Time Shifting

A time-shift of n_0 in $x[n]$ causes a multiplication of z^{-n_0} in $X(z)$

$$x[n - n_0] \leftrightarrow z^{-n_0} X(z) \tag{5.21}$$

The ROC for $x[n - n_0]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z = 0$ or $z = \infty$

Example 5.8

Find the z transform of $x[n]$ which has the form of:

$$x[n] = a^{n-1}u[n-1]$$

Employing the time-shifting property with $n_0 = 1$ and:

$$a^n u[n] \leftrightarrow \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

we easily obtain

$$a^{n-1}u[n-1] \leftrightarrow z^{-1} \cdot \frac{1}{1 - az^{-1}} = \frac{z^{-1}}{1 - az^{-1}}, \quad |z| > |a|$$

Note that using (5.1) with $|z| > |a|$ also produces the same result but this approach is less efficient:

$$X(z) = \sum_{n=1}^{\infty} a^{n-1} z^{-n} = a^{-1} \sum_{n=1}^{\infty} \left(az^{-1}\right)^n = a^{-1} \frac{az^{-1}\left[1 - \left(az^{-1}\right)^{\infty}\right]}{1 - az^{-1}} = \frac{z^{-1}}{1 - az^{-1}}$$

3. Multiplication by an Exponential Sequence (Modulation)

If we multiply $x[n]$ by z_0^n in the time domain, the variable z will be changed to z/z_0 in the z transform domain. That is:

$$z_0^n x[n] \leftrightarrow X(z/z_0) \qquad (5.22)$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, the ROC for $z_0^n x[n]$ is $|z_0|R_+ < |z| < |z_0|R_-$

<u>Example 5.9</u>

With the use of the following z transform pair:

$$u[n] \leftrightarrow \frac{1}{1-z^{-1}}, \quad |z| > 1$$

Find the z transform of $x[n]$ which has the form of:

$$x[n] = a^n \cos(bn) u[n]$$

Noting that $\cos(bn) = (e^{jbn} + e^{-jbn})/2$, $x[n]$ can be written as:

$$x[n] = \frac{1}{2}\left(ae^{jb}\right)^n u[n] + \frac{1}{2}\left(ae^{-jb}\right)^n u[n]$$

By means of the modulation property of (5.22) with the substitution of $z_0 = ae^{jb}$ and $z_0 = ae^{-jb}$, we obtain:

$$\frac{1}{2}\left(ae^{jb}\right)^n u[n] \leftrightarrow \frac{1}{2}\frac{1}{1-(z/(ae^{jb}))^{-1}} = \frac{1}{2}\frac{1}{1-ae^{jb}z^{-1}}, \quad |z| > |a|$$

and

$$\frac{1}{2}\left(ae^{-jb}\right)^n u[n] \leftrightarrow \frac{1}{2}\frac{1}{1-(z/(ae^{-jb}))^{-1}} = \frac{1}{2}\frac{1}{1-ae^{-jb}z^{-1}}, \quad |z| > |a|$$

By means of the linearity property, it follows that

$$X(z) = \frac{1}{2}\frac{1}{1-ae^{jb}z^{-1}} + \frac{1}{2}\frac{1}{1-ae^{-jb}z^{-1}} = \frac{1-a\cos(b)z^{-1}}{1-2a\cos(b)z^{-1}+a^2z^{-2}}, |z| > |a|$$

which agrees with Table 5.1.

4. Differentiation

Differentiating $X(z)$ with respect to z corresponds to multiplying $x[n]$ by n in the time domain:

$$nx[n] \leftrightarrow -z\frac{dX(z)}{dz} \tag{5.23}$$

The ROC for $nx[n]$ is basically identical to that of $X(z)$ except for the possible addition or deletion of $z=0$ or $z=\infty$

<u>Example 5.10</u>
Determine the z transform of $x[n]=na^n u[n]$.

Since

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z|>|a|$$

and

$$\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right)=\frac{d\left(1-az^{-1}\right)^{-1}}{d\left(1-az^{-1}\right)}\cdot\frac{d\left(1-az^{-1}\right)}{dz}=-\frac{az^{-2}}{\left(1-az^{-1}\right)^2}$$

By means of the differentiation property, we have

$$na^n u[n] \leftrightarrow -z\cdot -\frac{az^{-2}}{\left(1-az^{-1}\right)^2}=\frac{az^{-1}}{\left(1-az^{-1}\right)^2}, \quad |z|>|a|$$

which agrees with Table 5.1.

5. Conjugation

The z transform pair for $x^*[n]$ is:

$$x^*[n] \leftrightarrow X^*(z^*) \tag{5.24}$$

The ROC for $x^*[n]$ is identical to that of $x[n]$

6. Time Reversal

The z transform pair for $x[-n]$ is:

$$x[-n] \leftrightarrow X(z^{-1}) \tag{5.25}$$

If the ROC for $x[n]$ is $R_+ < |z| < R_-$, the ROC for $x[-n]$ is $1/R_- < |z| < 1/R_+$

<u>Example 5.11</u>

Determine the z transform of $x[n] = -na^{-n}u[-n]$

Using Example 5.10:

$$na^n u[n] \leftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}, \quad |z| > |a|$$

and from the time reversal property:

$$X(z) = \frac{az}{(1-az)^2} = \frac{a^{-1}z^{-1}}{(1-a^{-1}z^{-1})^2}, \quad |z| < |a^{-1}|$$

7. Convolution

Let $(x_1[n], X_1(z))$ and $(x_2[n], X_2(z))$ be two z transform pairs with ROCs $\mathcal{R}_{x_1}$ and $\mathcal{R}_{x_2}$, respectively. Then we have:

$$x_1[n] \otimes x_2[n] \leftrightarrow X_1(z)X_2(z) \qquad (5.26)$$

and its ROC includes $\mathcal{R}_{x_1} \cap \mathcal{R}_{x_2}$.

The proof is given as follows.

Let

$$y[n] = x_1[n] \otimes x_2[n] = \sum_{k=-\infty}^{\infty} x_1[k]x_2[n-k] \qquad (5.27)$$

With the use of the time shifting property, $Y(z)$ is:

$$
\begin{aligned}
Y(z) &= \sum_{n=-\infty}^{\infty} \left[\sum_{k=-\infty}^{\infty} x_1[k] x_2[n-k] \right] z^{-n} \\
&= \sum_{k=-\infty}^{\infty} x_1[k] \left[\sum_{n=-\infty}^{\infty} x_2[n-k] z^{-n} \right] \\
&= \sum_{k=-\infty}^{\infty} x_1[k] X_2(z) z^{-k} \\
&= X_1(z) X_2(z)
\end{aligned}
\qquad (5.28)
$$

Inverse z Transform

Inverse z transform corresponds to finding $x[n]$ given $X(z)$ and its ROC

The z transform and inverse z transform are one-to-one mapping provided that the ROC is given:

$$x[n] \leftrightarrow X(z) \tag{5.29}$$

There are 4 commonly used techniques to evaluate the inverse z transform. They are

1. Inspection
2. Partial Fraction Expansion
3. Power Series Expansion
4. Cauchy Integral Theorem

Inspection

When we are familiar with certain transform pairs, we can do the inverse z transform by inspection

Example 5.12

Determine the inverse z transform of $X(z)$ which is expressed as:

$$X(z) = \frac{z}{2z-1}, \quad |z| > 0.5$$

We first rewrite $X(z)$ as:

$$X(z) = \frac{0.5}{1-0.5z^{-1}}$$

Making use of the following transform pair in Table 5.1:

$$a^n u[n] \leftrightarrow \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

and putting $a=0.5$, we have:

$$\frac{0.5}{1-0.5z^{-1}} \leftrightarrow 0.5(0.5)^n u[n]$$

By inspection, the inverse z transform is:

$$x[n] = (0.5)^{n+1} u[n]$$

Partial Fraction Expansion

It is useful when $X(z)$ is a rational function in z^{-1}:

$$X(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{5.30}$$

For pole and zero determination, it is advantageous to multiply z^{M+N} to both numerator and denominator:

$$X(z) = \frac{z^N \sum_{k=0}^{M} b_k z^{M-k}}{z^M \sum_{k=0}^{N} a_k z^{N-k}} \tag{5.31}$$

- When $M > N$, there are $(M-N)$ pole(s) at $z=0$
- When $M < N$, there are $(N-M)$ zero(s) at $z=0$

To obtain the partial fraction expansion from (5.30), the first step is to determine the N nonzero poles, $c_1, c_2, \cdots, c_N$

There are 4 cases to be considered:

Case 1: $M < N$ and all poles are of first order

For first-order poles, all $\{c_k\}$ are distinct. $X(z)$ is:

$$X(z) = \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \tag{5.32}$$

For each first-order term of $A_k / \left(1 - c_k z^{-1}\right)$, its inverse z transform can be easily obtained by inspection

Multiplying both sides by $\left(1-c_k z^{-1}\right)$ and evaluating for $z=c_k$

$$A_k = \left(1-c_k z^{-1}\right) X(z)\Big|_{z=c_k} \quad (5.33)$$

An illustration for computing A_1 with $N=2>M$ is:

$$X(z) = \frac{A_1}{1-c_1 z^{-1}} + \frac{A_2}{1-c_2 z^{-1}}$$
$$\Rightarrow \left(1-c_1 z^{-1}\right) X(z) = A_1 + \frac{A_2\left(1-c_1 z^{-1}\right)}{1-c_2 z^{-1}} \quad (5.34)$$

Substituting $z=c_1$, we get A_1

In summary, three steps are:

- Find poles
- Find $\{A_k\}$
- Perform inverse z transform for the fractions by inspection

Example 5.13

Find the pole and zero locations of $H(z)$:

$$H(z) = -\frac{1 + 0.1z^{-1}}{1 - 2.05z^{-1} + z^{-2}}$$

Then determine the inverse z transform of $H(z)$.

We first multiply z^2 to both numerator and denominator polynomials to obtain:

$$H(z) = -\frac{z\left(z + 0.1\right)}{z^2 - 2.05z + 1}$$

Apparently, there are two zeros at $z = 0$ and $z = -0.1$. On the other hand, by solving the quadratic equation at the denominator polynomial, the poles are determined as $z = 0.8$ and $z = 1.25$.

According to (5.32), we have:

$$H(z) = \frac{A_1}{1 - 0.8z^{-1}} + \frac{A_2}{1 - 1.25z^{-1}}$$

Employing (5.33), A_1 is calculated as:

$$A_1 = \left(1 - 0.8z^{-1}\right) H(z)\Big|_{z=0.8} = - \left.\frac{1 + 0.1z^{-1}}{1 - 1.25z^{-1}}\right|_{z=0.8} = 2$$

Similarly, A_2 is found to be -3. As a result, the partial fraction expansion for $H(z)$ is

$$H(z) = \frac{2}{1 - 0.8z^{-1}} - \frac{3}{1 - 1.25z^{-1}}$$

As the ROC is not specified, we investigate all possible scenarios, namely, $|z| > 1.25$, $0.8 < |z| < 1.25$, and $|z| < 0.8$.

For $|z| > 1.25$, we notice that

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| > 0.8$$

and

$$(1.25)^n u[n] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| > 1.25$$

where both ROCs agree with $|z| > 1.25$. Combining the results, the inverse z transform $h[n]$ is:

$$h[n] = \left(2(0.8)^n - 3(1.25)^n\right) u[n]$$

which is a right-sided sequence and aligns with P5.

For $0.8 < |z| < 1.25$, we make use of

$$(0.8)^n u[n] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| > 0.8$$

and

$$-(1.25)^n u[-n-1] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| < 1.25$$

where both ROCs agree with $0.8 < |z| < 1.25$. This implies:

$$h[n] = 2(0.8)^n u[n] + 3(1.25)^n u[-n-1]$$

which is a two-sided sequence and aligns with P7.

Finally, for $|z| < 0.8$:

$$-(0.8)^n u[-n-1] \leftrightarrow \frac{1}{1-0.8z^{-1}}, \quad |z| < 0.8$$

and

$$-(1.25)^n u[-n-1] \leftrightarrow \frac{1}{1-1.25z^{-1}}, \quad |z| < 1.25$$

where both ROCs agree with $|z| < 0.8$. As a result, we have:

$$h[n] = \left(-2(0.8)^n + 3(1.25)^n\right) u[-n-1]$$

which is a left-sided sequence and aligns with P6.

Suppose $h[n]$ is the impulse response of a discrete-time LTI system. Recall (3.15) and (3.16):

$$h[n] = 0, \quad n < 0$$

and

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

The three possible impulse responses:

- $h[n] = \left(2(0.8)^n - (1.25)^n\right) u[n]$ is the impulse response of a causal but unstable system
- $h[n] = 2(0.8)^n u[n] + (1.25)^n u[-n-1]$ corresponds to a noncausal but stable system
- $h[n] = \left(-2(0.8)^n + (1.25)^n\right) u[-n-1]$ is noncausal and unstable

Which of the h[n] has/have DTFT?

Case 2: $M \geq N$ and all poles are of first order

In this case, $X(z)$ can be expressed as:

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1}^{N} \frac{A_k}{1 - c_k z^{-1}} \qquad (5.35)$$

- B_l are obtained by long division of the numerator by the denominator, with the division process terminating when the remainder is of lower degree than the denominator
- A_k can be obtained using (5.33).

<u>Example 5.14</u>

Determine $x[n]$ which has z transform of the form:

$$X(z) = \frac{4 - 2z^{-1} + z^{-2}}{1 - 1.5z^{-1} + 0.5z^{-2}}, \quad |z| > 1$$

The poles are easily determined as $z=0.5$ and $z=1$

According to (5.35) with $M=N=2$:

$$X(z)=B_0+\frac{A_1}{1-0.5z^{-1}}+\frac{A_2}{1-z^{-1}}$$

The value of B_0 is found by dividing the numerator polynomial by the denominator polynomial as follows:

$$\begin{array}{r}
2\phantom{+2z^{-1}+4} \\
0.5z^{-2}-1.5z^{-1}+1\overline{\big)\, z^{-2}-2z^{-1}+4} \\
\underline{z^{-2}-3z^{-1}+2} \\
z^{-1}+2
\end{array}$$

That is, $B_0=2$. Thus $X(z)$ is expressed as

$$X(z)=2+\frac{2+z^{-1}}{(1-0.5z^{-1})(1-z^{-1})}=2+\frac{A_1}{1-0.5z^{-1}}+\frac{A_2}{1-z^{-1}}$$

According to (5.33), A_1 and A_2 are calculated as

$$A_1 = \left.\frac{4 - 2z^{-1} + z^{-2}}{1 - z^{-1}}\right|_{z=0.5} = -4$$

and

$$A_2 = \left.\frac{4 - 2z^{-1} + z^{-2}}{1 - 0.5z^{-1}}\right|_{z=1} = 6$$

With $|z| > 1$:

$$\delta[n] \leftrightarrow 1$$

$$(0.5)^n u[n] \leftrightarrow \frac{1}{1 - 0.5z^{-1}}, \quad |z| > 0.5$$

and

$$u[n] \leftrightarrow \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

the inverse z transform $x[n]$ is:

$$x[n] = 2\delta[n] - 4(0.5)^n u[n] + 6u[n]$$

Case 3: $M < N$ with multiple-order pole(s)

If $X(z)$ has a s-order pole at $z = c_i$ with $s \geq 2$, this means that there are s repeated poles with the same value of c_i. $X(z)$ is:

$$X(z) = \sum_{k=1,k\neq i}^{N} \frac{A_k}{1 - c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1 - c_i z^{-1}\right)^m} \qquad (5.36)$$

- When there are two or more multiple-order poles, we include a component like the second term for each corresponding pole

- A_k can be computed according to (5.33)

- C_m can be calculated from:

$$C_m = \frac{1}{(s-m)!(-c_i)^{s-m}} \cdot \frac{d^{s-m}}{dw^{s-m}} \left[(1 - c_i w)^s X(w^{-1}) \right] \Big|_{w=c_i^{-1}} \qquad (5.37)$$

Example 5.15

Determine the partial fraction expansion for $X(z)$:

$$X(z)=\frac{4}{\left(1+z^{-1}\right)\left(1-z^{-1}\right)^2}$$

It is clear that $X(z)$ corresponds to Case 3 with $N=3>M$ and one second-order pole at $z=1$. Hence $X(z)$ is:

$$X(z)=\frac{A_1}{1+z^{-1}}+\frac{C_1}{1-z^{-1}}+\frac{C_2}{\left(1-z^{-1}\right)^2}$$

Employing (5.33), A_1 is:

$$A_1=\left.\frac{4}{\left(1-z^{-1}\right)^2}\right|_{z=-1}=1$$

Applying (5.37), C_1 is:

$$
\begin{aligned}
C_1 &= \frac{1}{(2-1)!(-1)^{2-1}} \cdot \frac{d}{dw}\left[(1-1\cdot w)^2 \frac{4}{(1+w)(1-w)^2}\right]\bigg|_{w=1} \\
&= -\frac{d}{dw}\frac{4}{1+w}\bigg|_{w=1} \\
&= \frac{4}{(1+w)^2}\bigg|_{w=1} \\
&= 1
\end{aligned}
$$

and

$$
\begin{aligned}
C_2 &= \frac{1}{(2-2)!(-1)^{2-2}} \cdot \left[(1-1\cdot w)^2 \frac{4}{(1+w)(1-w)^2}\right]\bigg|_{w=1} \\
&= \frac{4}{1+w}\bigg|_{w=1} \\
&= 2
\end{aligned}
$$

Therefore, the partial fraction expansion for $X(z)$ is

$$X(z) = \frac{1}{1+z^{-1}} + \frac{1}{1-z^{-1}} + \frac{2}{\left(1-z^{-1}\right)^2}$$

Case 4: $M \geq N$ with multiple-order pole(s)

This is the most general case and the partial fraction expansion of $X(z)$ is

$$X(z) = \sum_{l=0}^{M-N} B_l z^{-l} + \sum_{k=1, k\neq i}^{N} \frac{A_k}{1-c_k z^{-1}} + \sum_{m=1}^{s} \frac{C_m}{\left(1-c_i z^{-1}\right)^m} \quad (5.38)$$

assuming that there is only one multiple-order pole of order $s \geq 2$ at $z = c_i$. It is easily extended to the scenarios when there are two or more multiple-order poles as in Case 3. The A_k , B_l and C_m can be calculated as in Cases 1, 2 and 3

Power Series Expansion

When $X(z)$ is expanded as power series according to (5.1):

$$X(z) = \sum_{n=-\infty}^{\infty} x[n]z^{-n} = \cdots + x[-1]z^{1} + x[0] + x[1]z^{-1} + x[2]z^{-2} + \cdots \quad (5.39)$$

any particular value of $x[n]$ can be determined by finding the coefficient of the appropriate power of z^{-1}

Example 5.16

Determine $x[n]$ which has z transform of the form:

$$X(z) = 2z^2\left(1 - 0.5z^{-1}\right)\left(1 + z^{-1}\right)\left(1 - z^{-1}\right), \quad 0 < |z| < \infty$$

Expanding $X(z)$ yields

$$X(z) = 2z^2 - z - 2 + z^{-1}$$

From (5.39), $x[n]$ is deduced as:

$$x[n] = 2\delta[n+2] - \delta[n+1] - 2\delta[n] + \delta[n-1]$$

Example 5.17

Determine $x[n]$ whose z transform is given as:

$$X(z) = \log(1 + az^{-1}), \quad |z| > |a|$$

With the use of the power series expansion for $\log(1+\lambda)$:

$$\log(1+\lambda) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}\lambda^n}{n}, \quad |\lambda| < 1$$

$X(z)$ with $|az^{-1}| < 1$ can be expressed as

$$\log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} a^n z^{-n}}{n}$$

From (5.39), $x[n]$ is deduced as:

$$x[n] = \frac{(-1)^{n+1} a^n}{n} u[n-1]$$

Example 5.18

Determine $x[n]$ whose z transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| > |a|$$

With the use of

$$\frac{1}{1-\lambda} = 1 + \lambda + \lambda^2 + \cdots\ , \quad |\lambda| < 1$$

Carrying out long division in $X(z)$ with $|az^{-1}| < 1$:

$$X(z) = 1 + az^{-1} + \left(az^{-1}\right)^2 + \cdots$$

From (5.39), $x[n]$ is deduced as:

$$x[n] = a^n u[n]$$

which agrees with Example 5.1 and Table 5.1

Example 5.19

Determine $x[n]$ whose z transform has the form of:

$$X(z) = \frac{1}{1 - az^{-1}}, \quad |z| < |a|$$

We first express $X(z)$ as:

$$X(z) = \frac{-a^{-1}z}{-a^{-1}z} \cdot \frac{1}{1 - az^{-1}} = \frac{-a^{-1}z}{1 - a^{-1}z}$$

Carrying out long division in $X(z)$ with $|a^{-1}z| < 1$:

$$X(z) = -a^{-1}z \left(1 + a^{-1}z + \left(a^{-1}z\right)^2 + \cdots\right)$$

From (5.39), $x[n]$ is deduced as:

$$x[n] = -a^n u[-n-1]$$

which agrees with Example 5.2 and Table 5.1

Transfer Function of Linear Time-Invariant System

A LTI system can be characterized by the transfer function, which is a z transform expression

Starting with:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \qquad (5.40)$$

Applying z transform on (5.40) with the use of the linearity and time shifting properties, we have

$$Y(z) \sum_{k=0}^{N} a_k z^{-k} = X(z) \sum_{k=0}^{M} b_k z^{-k} \qquad (5.41)$$

The transfer function, denoted by $H(z)$, is defined as:

$$H(z)=\frac{Y(z)}{X(z)}=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \tag{5.42}$$

The system impulse response $h[n]$ is given by the inverse z transform of $H(z)$ with an appropriate ROC, that is, $h[n] \leftrightarrow H(z)$, such that $y[n]=x[n]\otimes h[n]$. This suggests that we can first take the z transforms for $x[n]$ and $h[n]$, then multiply $X(z)$ by $H(z)$, and finally perform the inverse z transform of $X(z)H(z)$.

Example 5.20

Determine the transfer function for a LTI system whose input $x[n]$ and output $y[n]$ are related by:

$$y[n] = 0.1y[n-1] + x[n] + x[n-1]$$

Applying z transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z)\left(1 - 0.1z^{-1}\right) = X(z)\left(1 + z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1 + z^{-1}}{1 - 0.1z^{-1}}$$

Note that there are two ROC possibilities, namely, $|z| > 0.1$ and $|z| < 0.1$ and we cannot uniquely determine $h[n]$

Example 5.21

Find the difference equation of a LTI system whose transfer function is given by

$$H(z) = \frac{\left(1+z^{-1}\right)\left(1-2z^{-1}\right)}{\left(1-0.5z^{-1}\right)\left(1+2z^{-1}\right)}$$

Let $H(z) = Y(z)/X(z)$. Performing cross-multiplication and inverse z transform, we obtain:

$$\begin{aligned}
&\left(1-0.5z^{-1}\right)\left(1+2z^{-1}\right)Y(z) = \left(1+z^{-1}\right)\left(1-2z^{-1}\right)X(z) \\
&\Rightarrow \left(1+1.5z^{-1}-z^{-2}\right)Y(z) = \left(1-z^{-1}-2z^{-2}\right)X(z) \\
&\Rightarrow y[n]+1.5y[n-1]-y[n-2] = x[n]-x[n-1]-2x[n-2]
\end{aligned}$$

Examples 5.20 and 5.21 imply the equivalence between the difference equation and transfer function

Example 5.22

Compute the impulse response $h[n]$ for a LTI system which is characterized by the following difference equation:

$$y[n] = x[n] - x[n-1]$$

Applying z transform on the difference equation with the use of the linearity and time shifting properties, $H(z)$ is:

$$Y(z) = X(z)\left(1 - z^{-1}\right) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 - z^{-1}$$

There is only one ROC possibility, namely, $|z| > 0$. Taking the inverse z transform on $H(z)$, we get:

$$h[n] = \delta[n] - \delta[n-1]$$

which agrees with Example 3.12

Example 5.23

Determine the output $y[n]$ if the input is $x[n]=u[n]$ and the LTI system impulse response is $h[n]=\delta[n]+0.5\delta[n-1]$

The z transforms for $x[n]$ and $h[n]$ are

$$X(z)=\frac{1}{1-z^{-1}}, \quad |z|>1 \quad \text{and} \quad H(z)=1+0.5z^{-1} \quad |z|>0$$

As a result, we have:

$$Y(z)=X(z)H(z)=\frac{1}{1-z^{-1}}+0.5\frac{z^{-1}}{1-z^{-1}}, \quad |z|>1$$

Taking the inverse z transform of $Y(z)$ with the use of the time shifting property yields:

$$y[n]=u[n]+0.5u[n-1]$$

which agrees with Example 3.8