Responses of Digital Filters

Chapter Intended Learning Outcomes:

(i) Understanding the relationships between impulse response, frequency response, difference equation and transfer function in characterizing a linear time-invariant system

(ii) Ability to identify infinite impulse response (IIR) and finite impulse response (FIR) filters. Note that a digital filter is system which processes discrete-time signals

(iii) Ability to compute system frequency response

LTI System Characterization

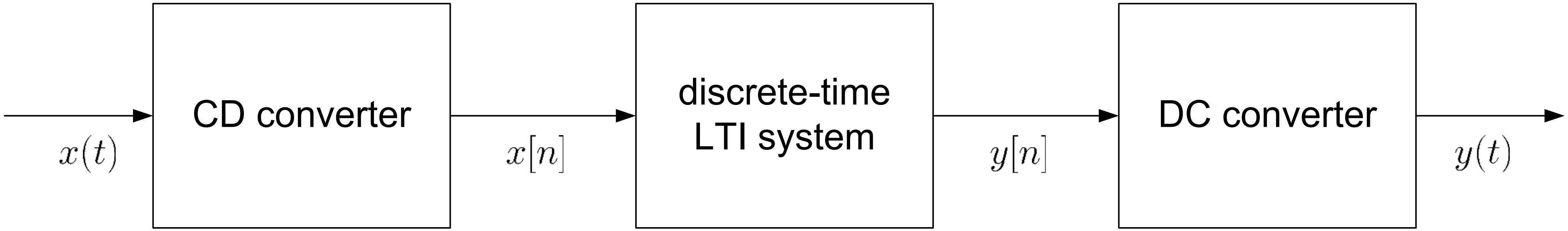

Fig.8.1: Processing of analog signal with LTI filter

The LTI system can be characterized as

- Impulse Response

Let $h[n]$ be the impulse response of the LTI filter. Recall from (3.19), it characterizes the system via the convolution:

$$y[n] = x[n] \otimes h[n] = \sum_{m=-\infty}^{\infty} x[m]h[n-m] \qquad (8.1)$$

$h[n]$ is the time-domain response of the LTI filter

- Frequency Response

We have from (6.17), which is the discrete-time Fourier transform (DTFT) of (8.1):

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) \tag{8.2}$$

$H(e^{j\omega})$ is the frequency-domain response of the LTI filter

- Difference Equation

A LTI system satisfies a difference equation of the form:

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \tag{8.3}$$

- Transfer Function

Taking the z transform on both sides of (8.3):

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}} \qquad (8.4)$$

Which of them can uniquely characterize a system? Which of them cannot?

Example 8.1

Given the difference equation with input $x[n]$ and output $y[n]$:

$$y[n] = ay[n-1] + x[n]$$

Find all possible ways to compute $y[n]$ given $x[n]$.

In fact, there are two ways to compute $y[n]$.

A straightforward and practical way is to implement a causal system by using the difference equation recursively with a given initial condition of $y[-1]$:

$$\begin{aligned} y[0] &= ay[-1] + x[0] \\ y[1] &= ay[0] + x[1] \\ y[2] &= ay[1] + x[2] \\ \cdots & \quad \cdots \quad \cdots \end{aligned}$$

On the other hand, it is possible to implement a noncausal system via reorganizing the difference equation as:

$$y[n] = ay[n-1] + x[n] \Rightarrow y[n-1] = \frac{1}{a}(y[n] - x[n])$$

In doing so, we need an initial future value of $y[-1]$ and future inputs, and the recursive implementation is:

$$\begin{aligned} y[-2] &= \frac{1}{a}(y[-1] - x[-1]) \\ y[-3] &= \frac{1}{a}(y[-2] - x[-2]) \\ y[-4] &= \frac{1}{a}(y[-3] - x[-3]) \\ \cdots & \quad \cdots \quad \cdots \end{aligned}$$

As a result, generally the difference equation cannot uniquely characterize the system.

Nevertheless, if causality is assumed, then the difference equation corresponds to a unique LTI system.

Alternatively, we can also study the computation of $y[n]$ using system transfer function $H(z)$:

$$Y(z) = az^{-1}Y(z) + X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1 - az^{-1}}$$

Since the ROC is not specified, there are two possible cases, namely, $|z| > |a|$ and $|z| < |a|$.

For $|z| > |a|$ or $|az^{-1}| < 1$, using inverse z transform:

$$\frac{Y(z)}{X(z)} = \frac{1}{1-az^{-1}} = 1 + az^{-1} + a^2z^{-2} + \cdots$$

$$\Rightarrow Y(z) = X(z)\left(1 + az^{-1} + a^2z^{-2} + \cdots\right)$$

$$\Rightarrow y[n] = x[n] + ax[n-1] + a^2x[n-2] + \cdots = \sum_{k=0}^{\infty} a^k x[n-k]$$

which corresponds to a causal system.

The same result can also be produced by first finding the system impulse response $h[n]$ via inverse z transform of $H(z)$:

$$h[n] = a^n u[n] \leftrightarrow H(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

and then computing $x[n] \otimes h[n]$.

For $|z| < |a|$ or $|a^{-1}z| < 1$, using inverse z transform:

$$\begin{aligned}\frac{Y(z)}{X(z)} &= \frac{1}{1-az^{-1}} = \frac{-a^{-1}z}{1-a^{-1}z} = -a^{-1}z\left(1+a^{-1}z+a^{-2}z^2+\cdots\right) \\ &\Rightarrow Y(z) = -X(z)\left(a^{-1}z+a^{-2}z^2+a^{-3}z^3+\cdots\right) \\ &\Rightarrow y[n] = -a^{-1}x[n+1]-a^{-2}x[n+2]-a^{-3}x[n+3]+\cdots \\ &= \sum_{k=-\infty}^{-1} -a^k x[n-k]\end{aligned}$$

which corresponds to a noncausal system.

<u>Example 8.2</u>

Discuss all possibilities for the LTI system whose input $x[n]$ and output $y[n]$ are related by:

$$y[n] - 5.4y[n-1] + 2y[n-2] = x[n]$$

Taking z transform on the difference equation yields:

$$H(z)=\frac{1}{1-5.4z^{-1}+2z^{-2}}=\frac{1}{\left(1-0.4z^{-1}\right)\left(1-5z^{-1}\right)}=\frac{z^2}{(z-0.4)(z-5)}$$

There are 3 possible choices:

$|z|>5$: The ROC is outside the circle with radius characterized by the largest-magnitude pole. The system can be causal but is not stable since the ROC does not include the unit circle

$0.4<|z|<5$: The system is stable because the ROC includes the unit circle but not causal

$|z|<0.4$: The system is unstable and noncausal

Impulse Response of Digital Filters

When $H(z)$ is a rational function of z^{-1} with only first-order poles, we have from (5.35):

$$H(z)=\frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}=\sum_{l=0}^{M-N} B_l z^{-l}+\sum_{k=1}^{N}\frac{A_k}{1-c_k z^{-1}} \qquad (8.5)$$

where the first component is present only if $M \geq N$.

If the system is causal, then the ROC must be of the form $|z| > |p_{\max}|$ where $p_{\max}$ is the largest-magnitude pole.

According to this ROC, the impulse response $h[n]$ is:

$$h[n] = \sum_{l=0}^{M-N} B_l \delta[n-l] + \sum_{k=1}^{N} A_k \left(c_k\right)^n u[n] \qquad (8.6)$$

There are two possible cases for (8.6) which correspond to IIR and FIR filters:

- IIR Filter

If $N \geq 1$ or there is at least one pole, the system is referred to as an IIR filter because $h[n]$ is of infinite duration.

- FIR Filter

If $N = 0$ or there is no pole, the system is referred to as a FIR filter because $h[n]$ is of finite duration.

Notice that the definitions of IIR and FIR systems also hold for noncausal systems.

<u>Example 8.3</u>

Determine if the following difference equations correspond to IIR or FIR systems. All systems are assumed causal.

(a) $y[n] = 0.1y[n-1] + x[n]$

(b) $y[n] = x[n] + 2x[n-1] + 3x[n-2]$

Taking the z transform on (a) yields

$$Y(z) = az^{-1}Y(z) + X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-0.1z^{-1}}$$

which has one pole. For causal system:

$$h[n] = (0.1)^n u[n] \leftrightarrow H(z) = \frac{1}{1 - 0.1z^{-1}}, \quad |z| > 0.1$$

which is a right-sided sequence and corresponds to an IIR system as $h[n]$ is of infinite duration.

Similarly, we have for (b):

$$Y(z) = X(z) + 2z^{-1}X(z) + 3z^{-2}X(z) \Rightarrow H(z) = \frac{Y(z)}{X(z)} = 1 + 2z^{-1} + 3z^{-2}$$

which does not have any nonzero pole. Hence

$$h[n] = \delta[n] + 2\delta[n-1] + 3\delta[n-2]$$

which corresponds to a FIR system as $h[n]$ is of finite duration.

Frequency Response of Digital Filters

The frequency response of a LTI system whose impulse response $h[n]$ is obtained by taking the DTFT of $h[n]$:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} \qquad (8.7)$$

According to (5.8), $H(e^{j\omega})$ is also obtained if $H(z)$ is available:

$$H(e^{j\omega}) = H(z)\big|_{z=e^{j\omega}} = \frac{\displaystyle\sum_{k=0}^{M} b_k e^{-j\omega k}}{\displaystyle\sum_{k=0}^{N} a_k e^{-\omega k}} \qquad (8.8)$$

assuming that the ROC of $H(z)$ includes the unit circle.

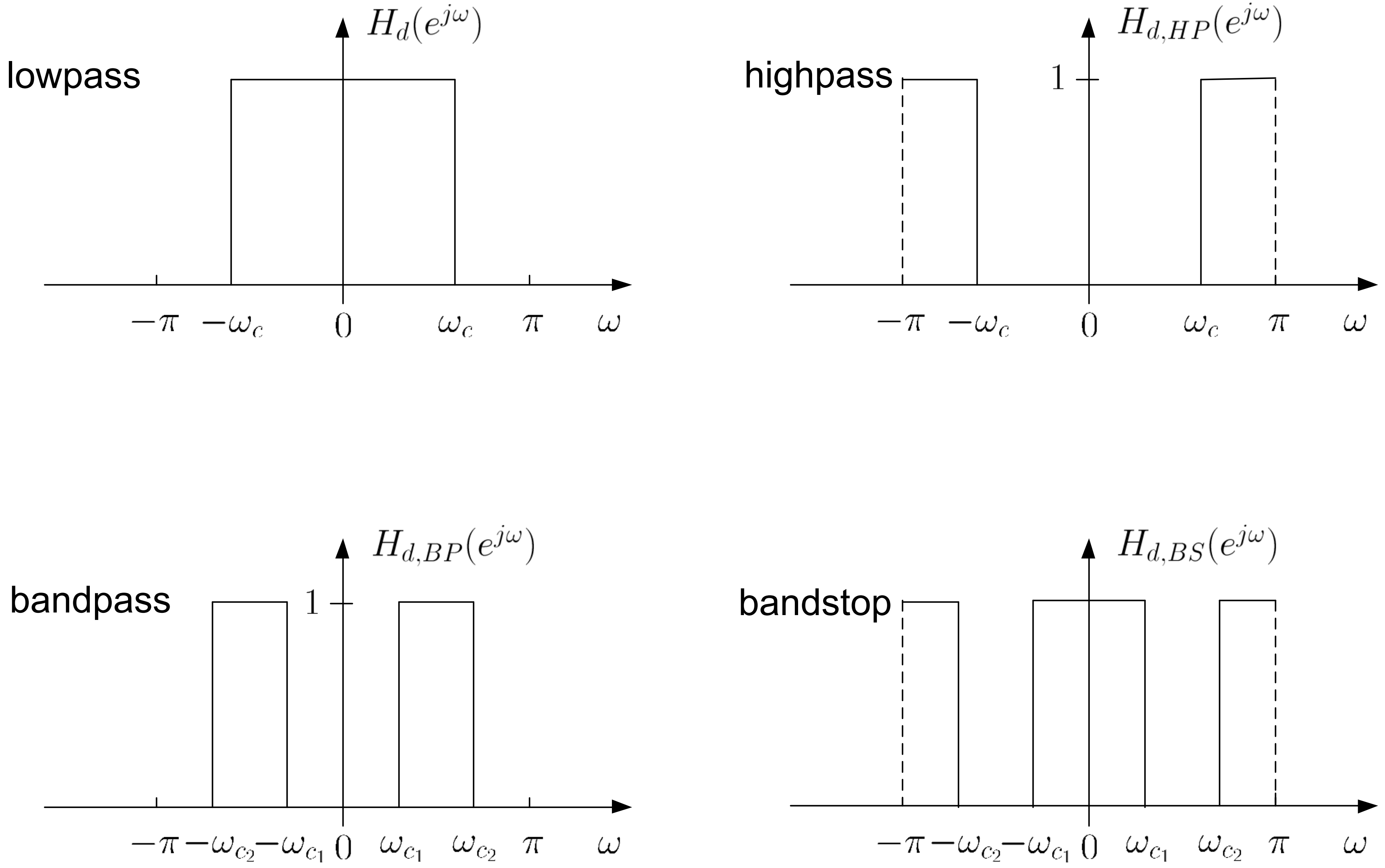

Fig.8.2: Typical filters described in frequency domain

Example 8.4

Plot the frequency response of the system with impulse response $h[n]$ of the form:

$$h[n] = \begin{cases} \text{sinc}(0.1(n-50)), & 0 \leq n \leq 100 \\ 0, & \text{otherwise} \end{cases}$$

where

$$\text{sinc}(u) = \frac{\sin(\pi u)}{\pi u}$$

Following Example 7.6, we append a large number of zeros at the end of $h[n]$ prior to performing discrete Fourier transform (DFT) to produce more DTFT samples.

Is it a lowpass filter? Why?

The MATLAB program is provided as `ex8_4.m`.

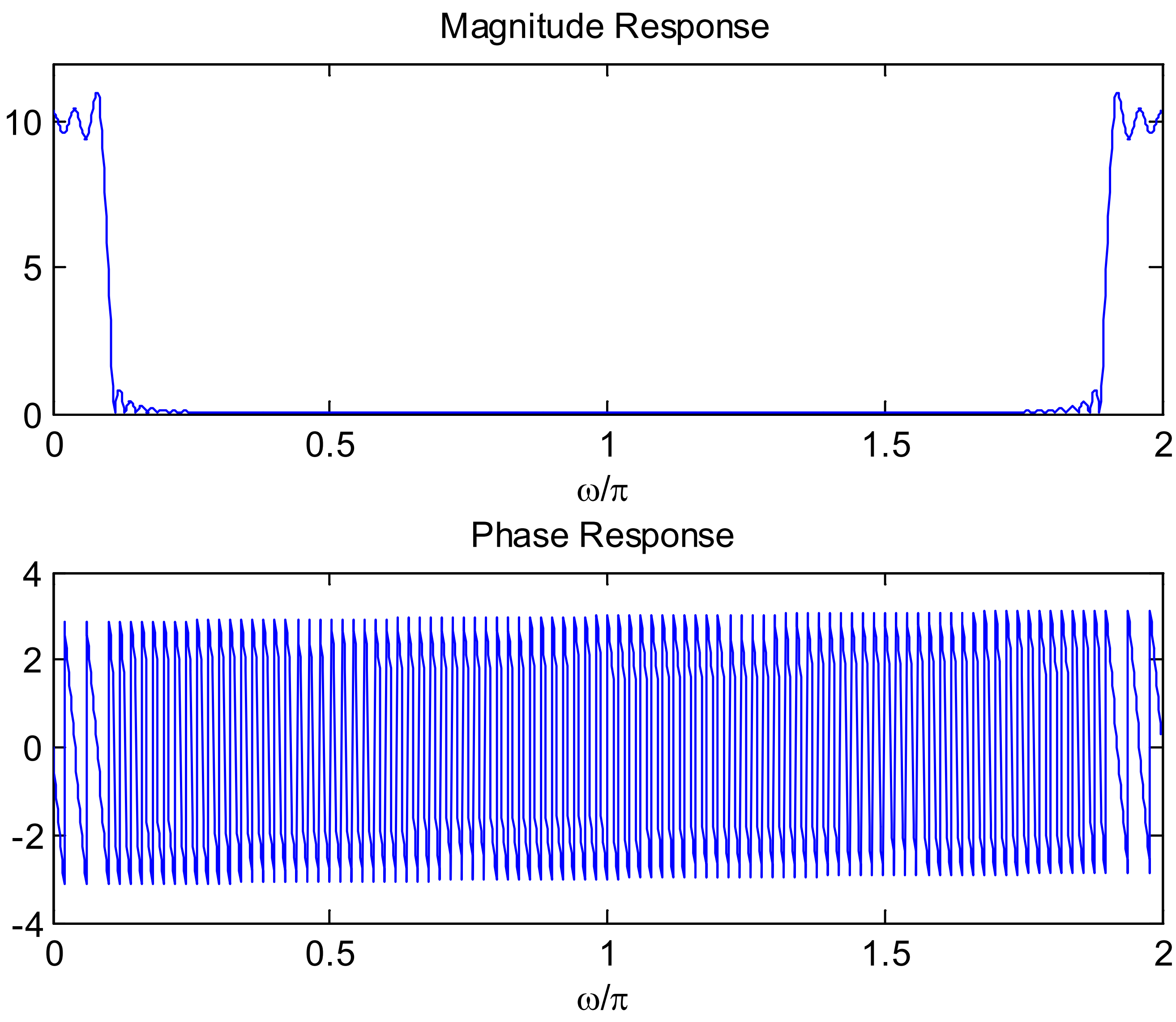

Fig.8.3: Frequency response for sinc $h[n]$

Example 8.5

Plot the frequency response of the system with transfer function $H(z)$ of the form:

$$H(z) = \frac{1 + 2z^{-1} + 3z^{-2}}{2 + 3z^{-1} + 4z^{-2}}$$

It is assumed that the ROC of $H(z)$ includes the unit circle.

The MATLAB code is

```
b=[1,2,3];
a=[2,3,4];
freqz(b,a);
```

Is it a lowpass filter? Why?

The MATLAB program is provided as `ex8_5.m`.

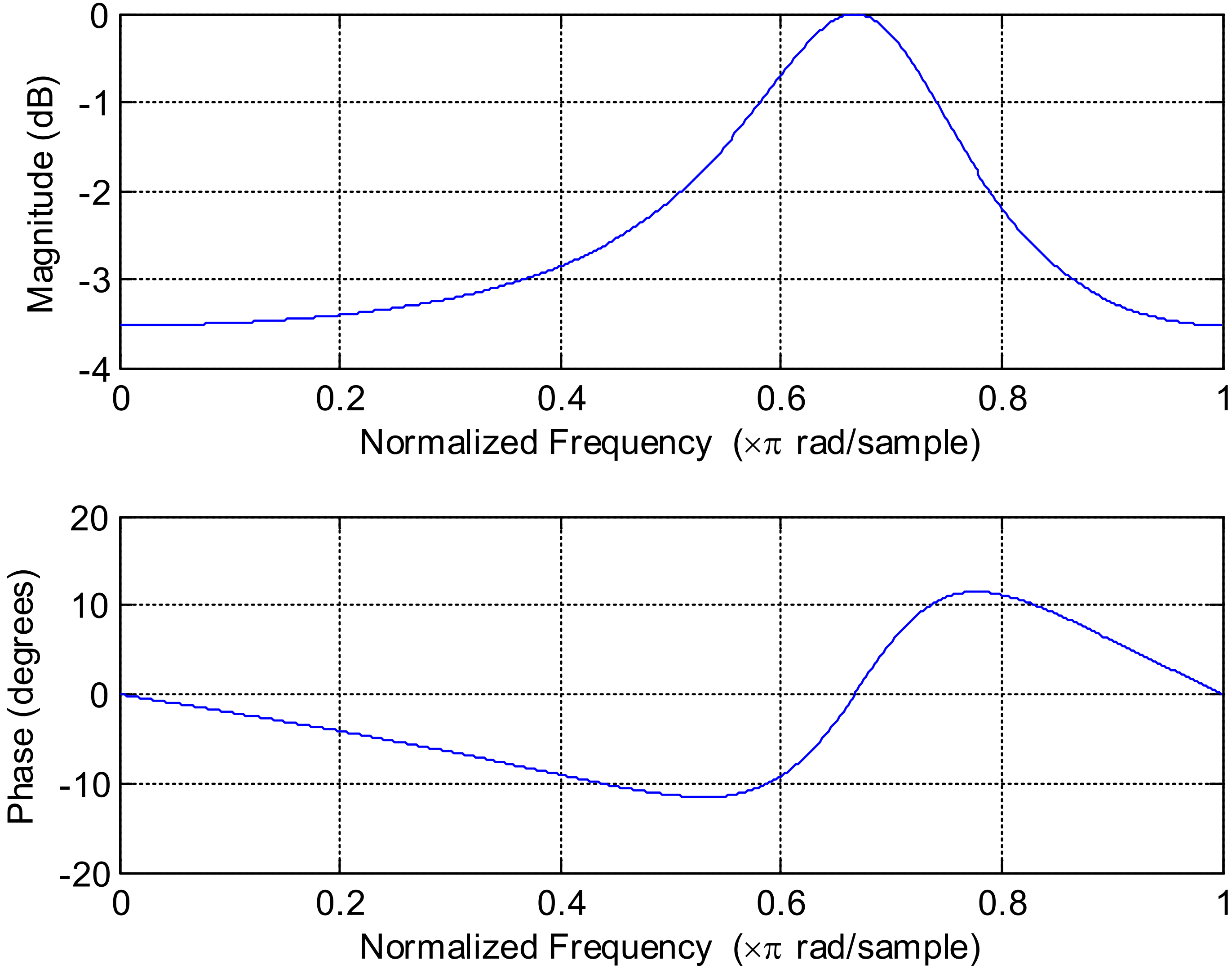

Fig.8.4: Frequency response for second-order system