Discrete Fourier Series & Discrete Fourier Transform

Chapter Intended Learning Outcomes

(i) Understanding the relationships between the z transform, discrete-time Fourier transform (DTFT), discrete Fourier series (DFS) and discrete Fourier transform (DFT)

(ii) Understanding the characteristics and properties of DFS and DFT

(iii) Ability to perform discrete-time signal conversion between the time and frequency domains using DFS and DFT and their inverse transforms

Discrete Fourier Series

DTFT may not be practical for analyzing $x[n]$ because $X(e^{j\omega})$ is a function of the continuous frequency variable ω and we cannot use a digital computer to calculate a continuum of functional values

DFS is a frequency analysis tool for periodic infinite-duration discrete-time signals which is practical because it is discrete in frequency

The DFS is derived from the Fourier series as follows.

Let $\tilde{x}[n]$ be a periodic sequence with fundamental period N where N is a positive integer. Analogous to (2.2), we have:

$$\tilde{x}[n] = \tilde{x}[n + rN] \quad (7.1)$$

for any integer value of r.

Let $x(t)$ be the continuous-time counterpart of $\tilde{x}[n]$. According to Fourier series expansion, $x(t)$ is:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\Omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kt}{T_p}} \quad (7.2)$$

which has frequency components at $\Omega = 0, \pm\Omega_0, \pm 2\Omega_0, \cdots$. Substituting $x(t) = \tilde{x}[n]$, $T_p = N$ and $t = n$:

$$\tilde{x}[n] = \sum_{k=-\infty}^{\infty} a_k e^{\frac{j2\pi kn}{N}} \quad (7.3)$$

Note that (7.3) is valid for discrete-time signals as only the sample points of $x(t)$ are considered.

It is seen that $\tilde{x}[n]$ has frequency components at $\omega = 0, \pm 2\pi/N, \pm(2\pi/N)(2), \cdots$, and the respective complex exponentials are $e^{j(2\pi/N(0))}, e^{\pm j(2\pi/N(1))}, e^{\pm j(2\pi/N(2))}, \cdots$.

Nevertheless, there are only N distinct frequencies in $\tilde{x}[n]$ due to the periodicity of $e^{j2\pi k/N}$.

Without loss of generality, we select the following N distinct complex exponentials, $e^{j(2\pi/N(0))}, e^{j(2\pi/N(1))}, \cdots, e^{j(2\pi/N(N-1))}$, and thus the infinite summation in (7.3) is reduced to:

$$\tilde{x}[n] = \sum_{k=0}^{N-1} a_k e^{\frac{j2\pi kn}{N}} \quad (7.4)$$

Defining $\tilde{X}[k] = N a_k$, $k = 0, 1, \cdots, N-1$, as the DFS coefficients, the inverse DFS formula is given as:

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] e^{\frac{j2\pi kn}{N}} \quad (7.5)$$

The formula for converting $\tilde{x}[n]$ to $\tilde{X}[k]$ is derived as follows.

Multiplying both sides of (7.5) by $e^{-j(2\pi/N)rn}$ and summing from $n=0$ to $n=N-1$:

$$\begin{aligned}\sum_{n=0}^{N-1}\tilde{x}[n]e^{\frac{-j2\pi rn}{N}} &= \sum_{n=0}^{N-1}\left(\frac{1}{N}\sum_{k=0}^{N-1}\tilde{X}[k]e^{\frac{j2\pi kn}{N}}\right)e^{\frac{-j2\pi rn}{N}} \\ &= \sum_{n=0}^{N-1}\frac{1}{N}\left(\sum_{k=0}^{N-1}\tilde{X}[k]e^{\frac{j2\pi(k-r)n}{N}}\right) \\ &= \sum_{k=0}^{N-1}\tilde{X}[k]\left[\frac{1}{N}\sum_{n=0}^{N-1}e^{\frac{j2\pi(k-r)n}{N}}\right] \qquad (7.6)\end{aligned}$$

Using the orthogonality identity of complex exponentials:

$$\frac{1}{N}\sum_{n=0}^{N-1}e^{\frac{j2\pi(k-r)n}{N}} = \begin{cases}1, & k-r=mN, \quad m \quad \text{is} \quad \text{an} \quad \text{integer} \\ 0, & \text{otherwise}\end{cases} \qquad (7.7)$$

(7.6) is reduced to

$$\sum_{n=0}^{N-1} \tilde{x}[n] e^{-\frac{j2\pi rn}{N}} = \tilde{X}[r] \quad (7.8)$$

which is also periodic with period N.

Let

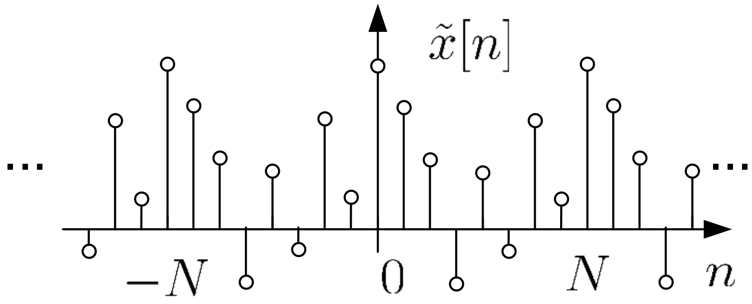
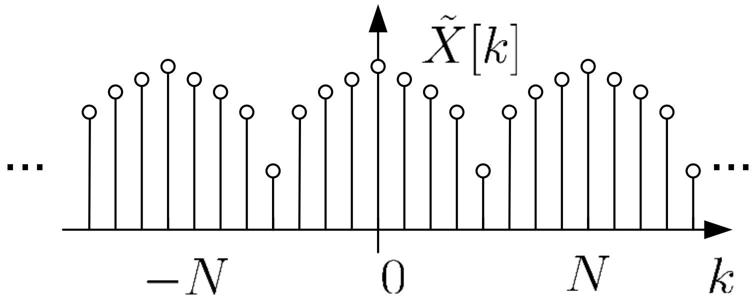
$$W_N = e^{-\frac{j2\pi}{N}} \quad (7.9)$$

The DFS analysis and synthesis pair can be written as:

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \quad (7.10)$$

and

$$\tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn} \quad (7.11)$$

<table>
<tr><th>time domain</th><th>frequency domain</th></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">

$$\tilde{X}[k] = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \Longrightarrow$$

$$\Longleftarrow \tilde{x}[n] = \frac{1}{N} \sum_{k=0}^{N-1} \tilde{X}[k] W_N^{-kn}$$

</td></tr>
<tr><td>discrete and periodic</td><td>discrete and periodic</td></tr>
</table>

Fig.7.1: Illustration of DFS

Example 7.1

Find the DFS coefficients of the periodic sequence $\tilde{x}[n]$ with a period of $N=5$. Plot the magnitudes and phases of $\tilde{X}[k]$. Within one period, $\tilde{x}[n]$ has the form of:

$$\tilde{x}[n] = \begin{cases} 1, \ n = 0,1,2 \\ 0, \ n = 3,4 \end{cases}$$

Using (7.10), we have

$$\begin{aligned} \tilde{X}[k] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{kn} \\ &= W_5^0 + W_5^k + W_5^{2k} \\ &= 1 + e^{-\frac{j2\pi k}{5}} + e^{-\frac{j4\pi k}{5}} \\ &= e^{-\frac{j2\pi k}{5}} \left(e^{\frac{j2\pi k}{5}} + 1 + e^{-\frac{j2\pi k}{5}} \right) \\ &= e^{-\frac{j2\pi k}{5}} \left[1 + 2\cos\left(\frac{2\pi k}{5} \right) \right] \end{aligned}$$

Similar to Example 6.2, we get:

$$|\tilde{X}[k]| = \left|1 + 2\cos\left(\frac{2\pi k}{5}\right)\right|$$

and

$$\angle(\tilde{X}[k]) = -\frac{2\pi k}{5} + \angle\left(1 + 2\cos\left(\frac{2\pi k}{5}\right)\right)$$

The key MATLAB code for plotting DFS coefficients is

```
N=5;
x=[1 1 1 0 0];
k=-N:2*N;                           %plot for 3 periods
Xm=abs(1+2.*cos(2*pi.*k/N));%magnitude computation
Xa=angle(exp(-2*j*pi.*k/5).*(1+2.*cos(2*pi.*k/N)));
                            %phase computation
```

The MATLAB program is provided as `ex7_1.m`.

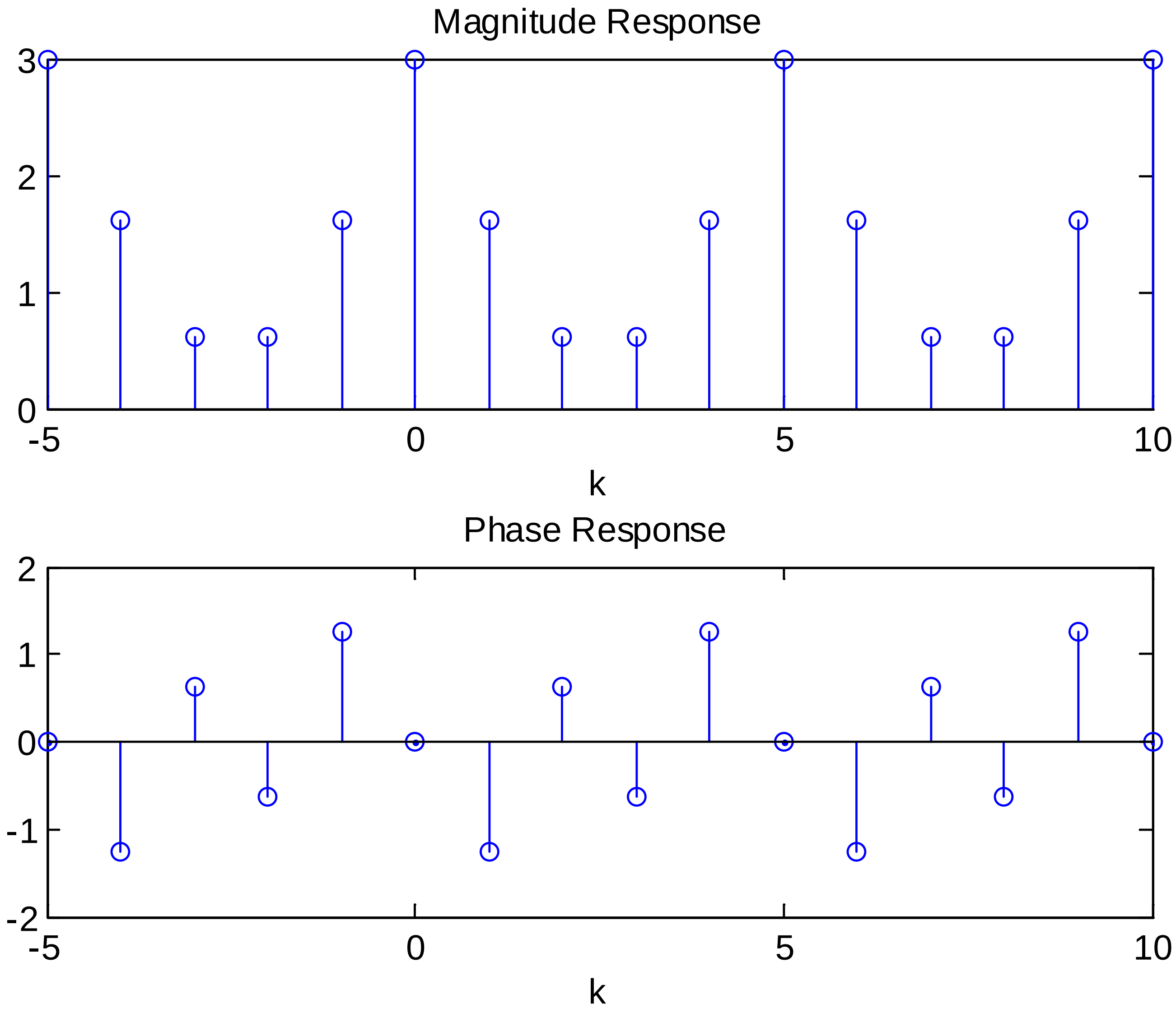


Fig.7.2: DFS plots

Relationship with DTFT

Let $x[n]$ be a finite-duration sequence which is extracted from a periodic sequence $\tilde{x}[n]$ of period N:

$$x[n] = \begin{cases} \tilde{x}[n], & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (7.12)$$

Recall (6.1), the DTFT of $x[n]$ is:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \quad (7.13)$$

With the use of (7.12), (7.13) becomes

$$X(e^{j\omega}) = \sum_{n=0}^{N-1} x[n] e^{-j\omega n} = \sum_{n=0}^{N-1} \tilde{x}[n] e^{-j\omega n} \quad (7.14)$$

Comparing the DFS and DTFT in (7.8) and (7.14), we have:

$$\tilde{X}[k] = X(e^{j\omega})\big|_{\omega=\frac{2\pi k}{N}} \qquad (7.15)$$

That is, $\tilde{X}[k]$ is equal to $X(e^{j\omega})$ sampled at N distinct frequencies between $\omega \in [0, 2\pi]$ with a uniform frequency spacing of $2\pi/N$.

Samples of $X(e^{j\omega})$ or DTFT of a finite-duration sequence $x[n]$ can be computed using the DFS of an infinite-duration periodic sequence $\tilde{x}[n]$, which is a periodic extension of $x[n]$.

Relationship with z Transform

$X(e^{j\omega})$ is also related to z transform of $x[n]$ according to (5.8):

$$X(e^{j\omega}) = X(z)\big|_{z=e^{j\omega}} \tag{7.16}$$

Combining (7.15) and (7.16), $\tilde{X}[k]$ is related to $X(z)$ as:

$$\tilde{X}[k] = X(z)\big|_{z=e^{\frac{j2\pi k}{N}}} = X(e^{\frac{j2\pi k}{N}}) \tag{7.17}$$

That is, $\tilde{X}[k]$ is equal to $X(z)$ evaluated at N equally-spaced points on the unit circle, namely, $1, e^{j2\pi/N}, \cdots, e^{j2(N-1)\pi/N}$.

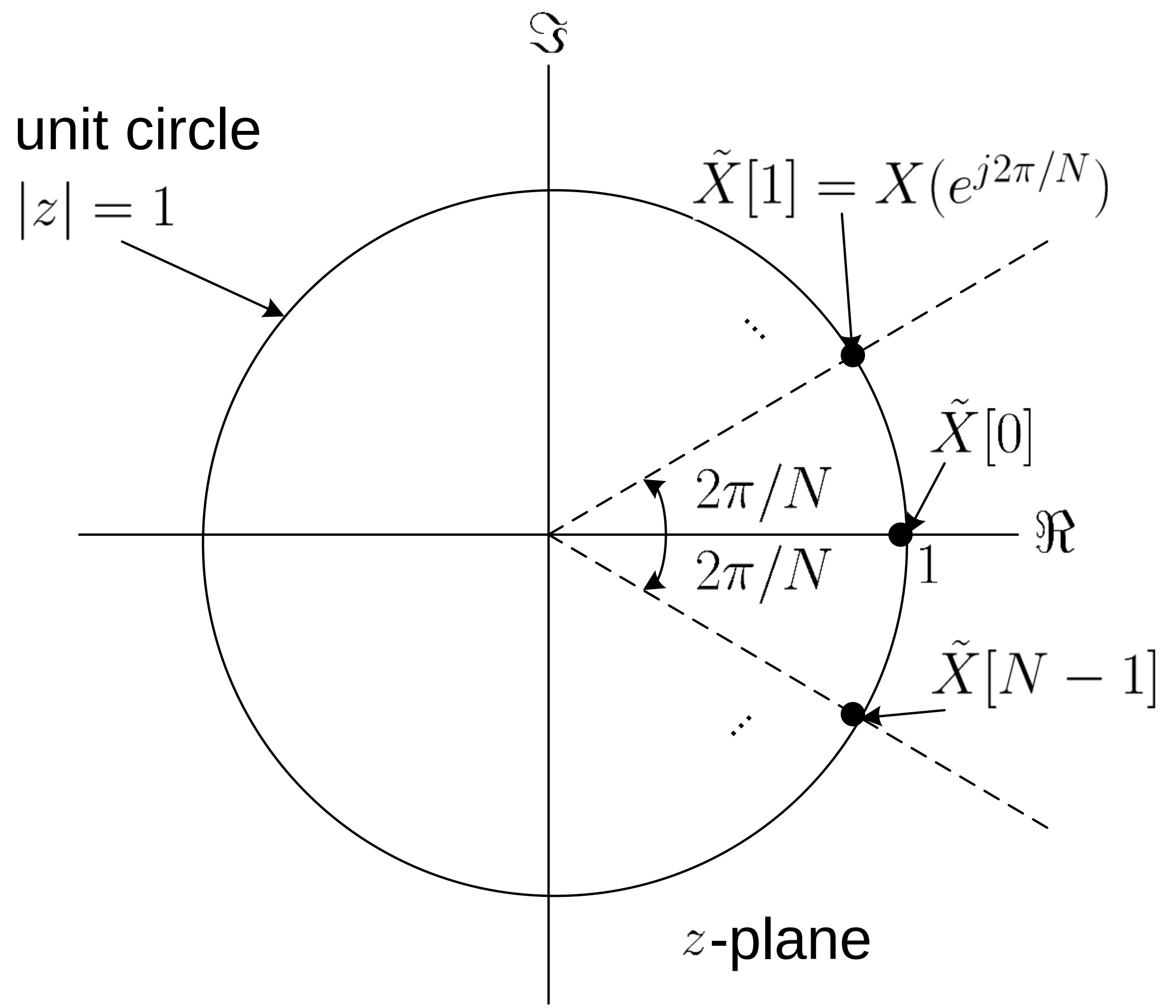


Fig.7.3: Relationship between $\tilde{X}[k]$, $X(e^{j\omega})$ and $X(z)$

Example 7.2

Determine the DTFT of a finite-duration sequence $x[n]$:

$$x[n] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Then compare the results with those in Example 7.1.

Using (6.1), the DTFT of $x[n]$ is computed as:

$$\begin{aligned} X(e^{j\omega}) &= \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \\ &= 1 + e^{-j\omega} + e^{-j2\omega} \\ &= e^{-j\omega}\left(e^{j\omega} + 1 + e^{-j\omega}\right) \\ &= e^{-j\omega}\left[1 + 2\cos(\omega)\right] \end{aligned}$$

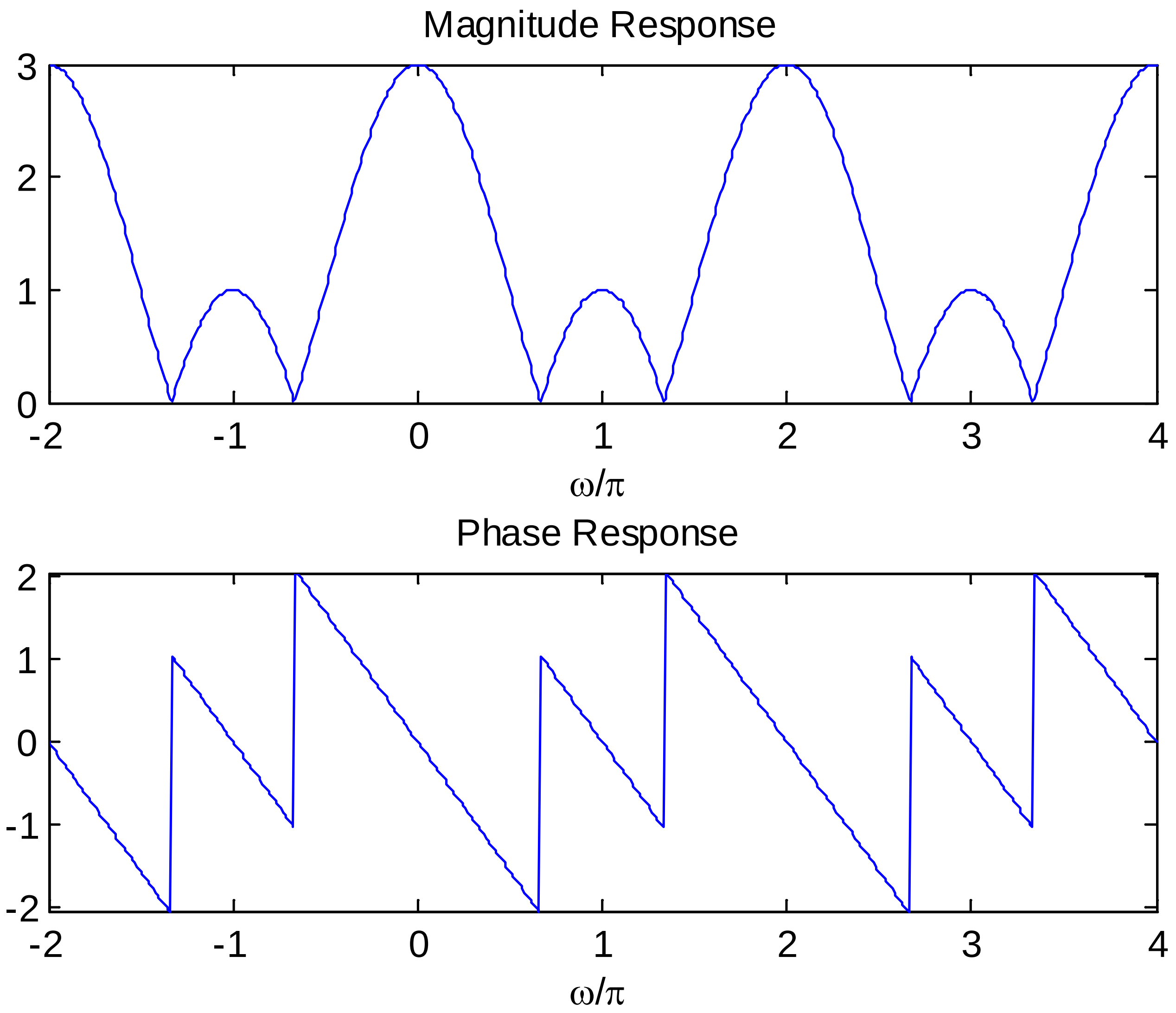


Fig.7.4: DTFT plots

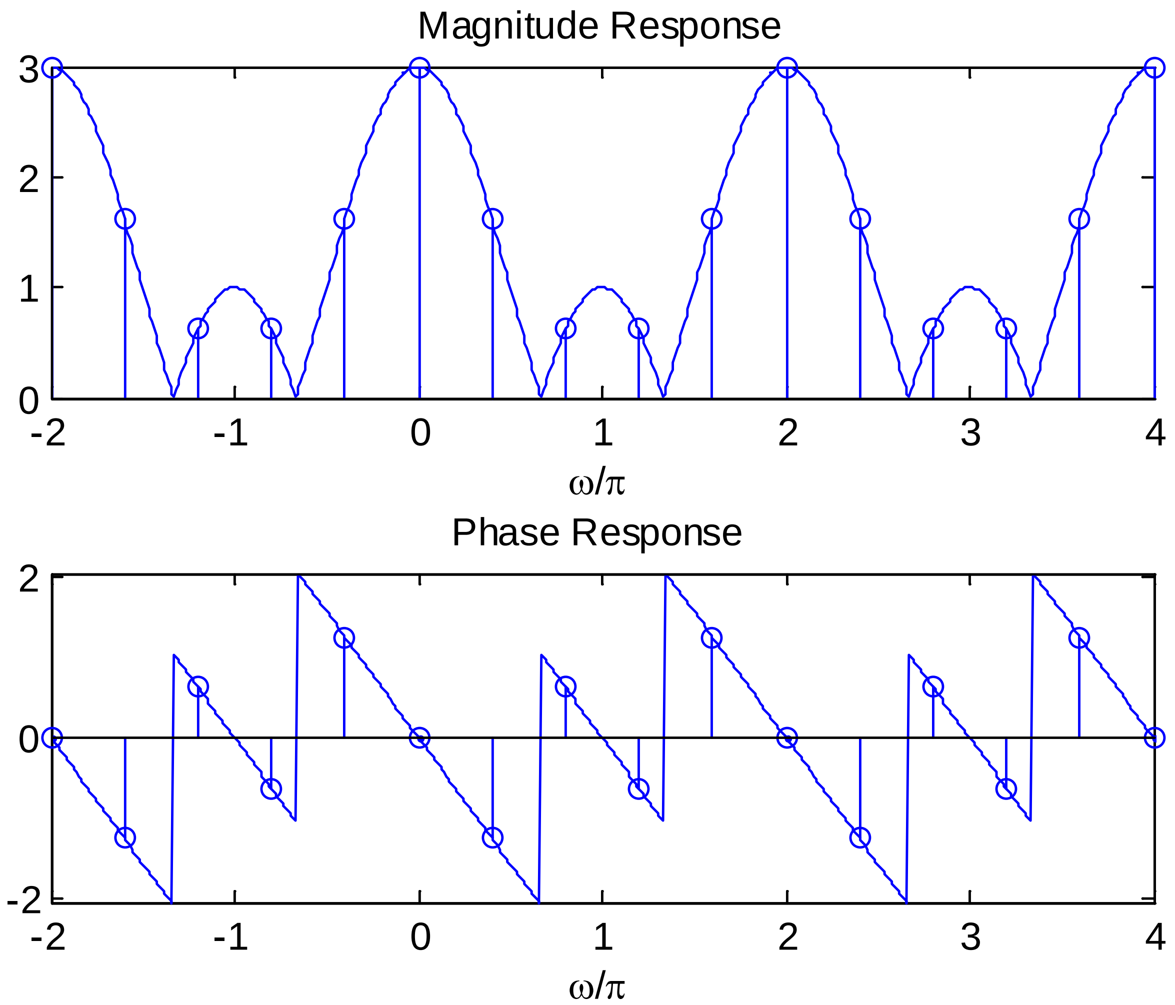


Fig.7.5: DFS and DTFT plots with $N = 5$

Suppose $\tilde{x}[n]$ in Example 7.1 is modified as:

$$\tilde{x}[n] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & n = 3, 4, \cdots, 9 \end{cases}$$

Via appending 5 zeros in each period, now we have $N = 10$.

What is the period of the DFS?

What is its relationship with that of Example 7.2?

How about if infinite zeros are appended?

The MATLAB programs are provided as `ex7_2.m`, `ex7_2_2.m` and `ex7_2_3.m`.

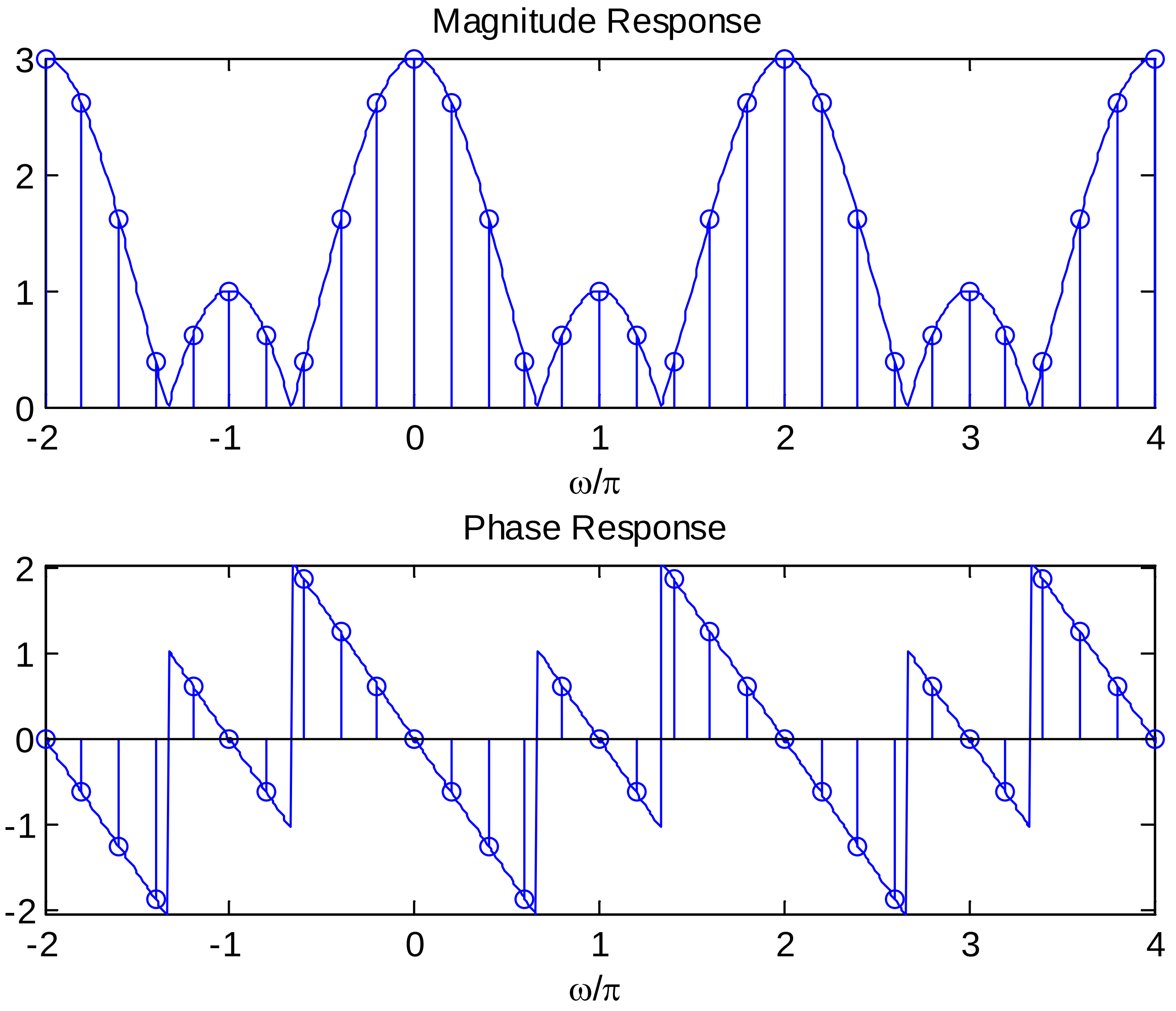


Fig.7.6: DFS and DTFT plots with $N = 10$

Properties of DFS

1. Periodicity

If $\tilde{x}[n]$ is a periodic sequence with period N, its DFS $\tilde{X}[k]$ is also periodic with period N:

$$\tilde{x}[n] = \tilde{x}[n + rN] \leftrightarrow \tilde{X}[k] = \tilde{X}[k + rN] \tag{7.18}$$

where r is any integer. The proof is obtained with the use of (7.10) and $W_N^{rN} = e^{-j2\pi r} = 1$ as follows:

$$\begin{aligned} \tilde{X}[k + rN] &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{(k+rN)n} = \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} W_N^{n(rN)} \\ &= \sum_{n=0}^{N-1} \tilde{x}[n] W_N^{nk} = \tilde{X}[k] \end{aligned} \tag{7.19}$$

2. Linearity

Let $(\tilde{x}_1[n], \tilde{X}_1[k])$ and $(\tilde{x}_2[n], \tilde{X}_2[k])$ be two DFS pairs with the same period of N. We have:

$$a\tilde{x}_1[n] + b\tilde{x}_2[n] \leftrightarrow a\tilde{X}_1[k] + b\tilde{X}_2[k] \quad (7.20)$$

3. Shift of Sequence

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then

$$\tilde{x}[n-m] \leftrightarrow W_N^{km}\tilde{X}[k] \quad (7.21)$$

and

$$W_N^{-nl}\tilde{x}[n] \leftrightarrow \tilde{X}[k-l] \quad (7.22)$$

where N is the period while m and l are any integers. Note that (7.21) follows (6.10) by putting $\omega = 2\pi k/N$ and (7.22) follows (6.11) via the substitution of $\omega_0 = 2\pi l/N$.

4. Duality

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then

$$\tilde{X}[n] \leftrightarrow N\tilde{x}[-k] \qquad (7.23)$$

5. Symmetry

If $\tilde{x}[n] \leftrightarrow \tilde{X}[k]$, then

$$\tilde{x}^*[n] \leftrightarrow \tilde{X}^*[-k] \qquad (7.24)$$

and

$$\tilde{x}^*[-n] \leftrightarrow \tilde{X}^*[k] \qquad (7.25)$$

Note that (7.24) corresponds to the DTFT conjugation property in (6.14) while (7.25) is similar to the time reversal property in (6.15).

6. Periodic Convolution

Let $(\tilde{x}_1[n], \tilde{X}_1[k])$ and $(\tilde{x}_2[n], \tilde{X}_2[k])$ be two DFS pairs with the same period of N. We have

$$\tilde{x}_1[n] \tilde{\otimes} \tilde{x}_2[n] = \sum_{m=0}^{N-1} \tilde{x}_1[m] \tilde{x}_2[n-m] \leftrightarrow \tilde{X}_1[k] \tilde{X}_2[k] \qquad (7.26)$$

Analogous to (6.18), $\tilde{\otimes}$ denotes discrete-time convolution within one period.

With the use of (7.11) and (7.21), the proof is given as follows:

$$
\begin{aligned}
\sum_{n=0}^{N-1}\left[\sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{x}_2[n-m]\right]W_N^{nk} &= \sum_{m=0}^{N-1}\tilde{x}_1[m]\left[\sum_{n=0}^{N-1}\tilde{x}_2[n-m]W_N^{nk}\right] \\
&= \sum_{m=0}^{N-1}\tilde{x}_1[m]\tilde{X}_2[k]W_N^{mk} \\
&= \tilde{X}_2[k]\left[\sum_{m=0}^{N-1}\tilde{x}_1[m]W_N^{mk}\right] \\
&= \tilde{X}_1[k]\tilde{X}_2[k] \qquad (7.27)
\end{aligned}
$$

To compute $\tilde{x}[n]\tilde{\otimes}\tilde{y}[n]$ where both $\tilde{x}[n]$ and $\tilde{y}[n]$ are of period N, we indeed only need the samples with $n = 0, 1, \cdots, N-1$.

Let $\tilde{z}[n] = \tilde{x}[n] \tilde{\otimes} \tilde{y}[n]$. Expanding (7.26), we have:

$$\tilde{z}[n] = \tilde{x}[0]\tilde{y}[n] + \cdots + \tilde{x}[N-2]\tilde{y}[n-(N-2)] + \tilde{x}[N-1]\tilde{y}[n-(N-1)] \quad (7.28)$$

For $n = 0$:

$$\begin{aligned} \tilde{z}[0] &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[0-(N-2)] + \tilde{x}[N-1]\tilde{y}[0-(N-1)] \\ &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[0-(N-2)+N] + \tilde{x}[N-1]\tilde{y}[0-(N-1)+N] \\ &= \tilde{x}[0]\tilde{y}[0] + \cdots + \tilde{x}[N-2]\tilde{y}[2] + \tilde{x}[N-1]\tilde{y}[1] \end{aligned} \quad (7.29)$$

For $n = 1$:

$$\begin{aligned} \tilde{z}[1] &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[1-(N-2)] + \tilde{x}[N-1]\tilde{y}[1-(N-1)] \\ &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[1-(N-2)+N] + \tilde{x}[N-1]\tilde{y}[1-(N-1)+N] \\ &= \tilde{x}[0]\tilde{y}[1] + \cdots + \tilde{x}[N-2]\tilde{y}[3] + \tilde{x}[N-1]\tilde{y}[2] \end{aligned} \quad (7.30)$$

A period of $\tilde{z}[n]$ can be computed in matrix form as:

$$\begin{bmatrix} \tilde{z}[0] \\ \tilde{z}[1] \\ \vdots \\ \tilde{z}[N-2] \\ \tilde{z}[N-1] \end{bmatrix} = \begin{bmatrix} \tilde{y}[0] & \tilde{y}[N-1] & \cdots & \tilde{y}[2] & \tilde{y}[1] \\ \tilde{y}[1] & \tilde{y}[0] & \cdots & \tilde{y}[3] & \tilde{y}[2] \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \tilde{y}[N-2] & \tilde{y}[N-3] & \cdots & \tilde{y}[0] & \tilde{y}[N-1] \\ \tilde{y}[N-1] & \tilde{y}[N-2] & \cdots & \tilde{y}[1] & \tilde{y}[0] \end{bmatrix} \begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \vdots \\ \tilde{x}[N-2] \\ \tilde{x}[N-1] \end{bmatrix} \quad (7.31)$$

Example 7.3

Given two periodic sequences $\tilde{x}[n]$ and $\tilde{y}[n]$ with period 4:

$$\begin{bmatrix} \tilde{x}[0] & \tilde{x}[1] & \tilde{x}[2] & \tilde{x}[3] \end{bmatrix} = \begin{bmatrix} 4 & -3 & 2 & -1 \end{bmatrix}$$

and

$$\begin{bmatrix} \tilde{y}[0] & \tilde{y}[1] & \tilde{y}[2] & \tilde{y}[3] \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix}$$

Compute $\tilde{z}[n] = \tilde{x}[n] \tilde{\otimes} \tilde{y}[n]$.

Using (7.31), $\tilde{z}[n]$ is computed as:

$$\begin{bmatrix} \tilde{z}[0] \\ \tilde{z}[1] \\ \tilde{z}[2] \\ \tilde{z}[3] \end{bmatrix} = \begin{bmatrix} \tilde{y}[0] & \tilde{y}[3] & \tilde{y}[2] & \tilde{y}[1] \\ \tilde{y}[1] & \tilde{y}[0] & \tilde{y}[3] & \tilde{y}[2] \\ \tilde{y}[2] & \tilde{y}[1] & \tilde{y}[0] & \tilde{y}[3] \\ \tilde{y}[3] & \tilde{y}[2] & \tilde{y}[1] & \tilde{y}[0] \end{bmatrix} \begin{bmatrix} \tilde{x}[0] \\ \tilde{x}[1] \\ \tilde{x}[2] \\ \tilde{x}[3] \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 4 & 3 & 2 \\ 2 & 1 & 4 & 3 \\ 3 & 2 & 1 & 4 \\ 4 & 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} -4 \\ 10 \\ 4 \\ 10 \end{bmatrix}$$

The square matrix can be determined using the MATLAB command `toeplitz([1,2,3,4],[1,4,3,2])`. That is, we only need to know its first row and first column.

Periodic convolution can be utilized to compute convolution of finite-duration sequences in (3.17) as follows.

Let $x[n]$ and $y[n]$ be finite-duration sequences with lengths M and N, respectively, and $z[n] = x[n] \otimes y[n]$ which has a length of $(M + N - 1)$

We append $(N - 1)$ and $(M - 1)$ zeros at the ends of $x[n]$ and $y[n]$ for constructing periodic $\tilde{x}[n]$ and $\tilde{y}[n]$ where both are of period $(M + N - 1)$

$z[n]$ is then obtained from one period of $\tilde{x}[n]\tilde{\otimes}\tilde{y}[n]$.

<u>Example 7.4</u>

Compute the convolution of $x[n]$ and $y[n]$ with the use of periodic convolution. The lengths of $x[n]$ and $y[n]$ are 2 and 3 with $x[0] = 2$, $x[1] = 3$, $y[0] = 1$, $y[1] = -4$ and $y[2] = 5$.

The length of $x[n] \otimes y[n]$ is 4. As a result, we append two zeros and one zero in $x[n]$ and $y[n]$, respectively. According to (7.31), the MATLAB code is:

```
toeplitz([1,-4,5,0],[1,0,5,-4])*[2;3;0;0]
```

which gives

```
2     -5     -2     15
```

Note that the command conv([2,3],[1,-4,5]) also produces the same result.

Discrete Fourier Transform

DFT is used for analyzing discrete-time finite-duration signals in the frequency domain

Let $x[n]$ be a finite-duration sequence of length N such that $x[n]=0$ outside $0 \leq n \leq N-1$. The DFT pair of $x[n]$ is:

$$X[k] = \begin{cases} \sum_{n=0}^{N-1} x[n] W_N^{kn}, & 0 \leq k \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (7.32)$$

and

$$x[n] = \begin{cases} \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}, & 0 \leq n \leq N-1 \\ 0, & \text{otherwise} \end{cases} \quad (7.33)$$

If we extend $x[n]$ to a periodic sequence $\tilde{x}[n]$ with period N, the DFS pair for $\tilde{x}[n]$ is given by (7.10)-(7.11). Comparing (7.32) and (7.10), $X[k]=\tilde{X}[k]$ for $0 \leq k \leq N-1$. As a result, DFT and DFS are equivalent within the interval of $[0, N-1]$

That is, we just extract one period of $\tilde{x}[n]$ and $\tilde{X}[k]$ to construct (7.32) and (7.33).

As a result, the DFT pair is not well theoretically justified and we cannot apply (7.32) to produce (7.33) or vice versa as in DFS, DTFT and Fourier transform.

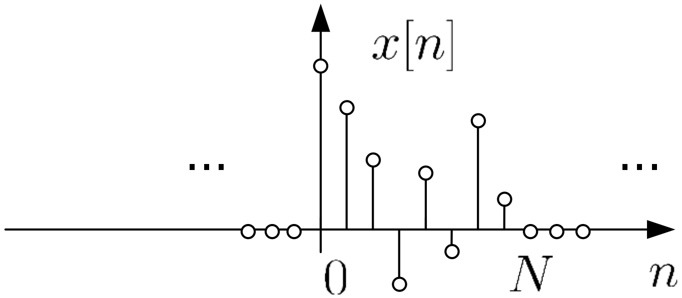
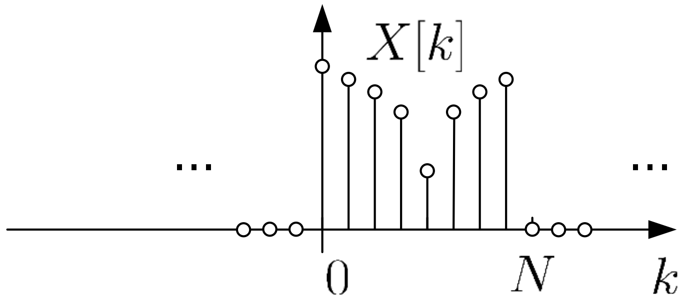
time domain	frequency domain
	
$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \Longrightarrow$	$\Longleftarrow x[n] = \frac{1}{N} \sum_{k=0}^{N-1} X[k] W_N^{-kn}$
discrete and finite	discrete and finite

Fig.7.7: Illustration of DFT

Example 7.5

Find the DFT coefficients of a finite-duration sequence $x[n]$ which has the form of

$$x[n] = \begin{cases} 1, & n = 0, 1, 2 \\ 0, & \text{otherwise} \end{cases}$$

Using (7.32) and Example 7.1 with $N = 3$, we have:

$$\begin{aligned} X[k] &= \sum_{n=0}^{2} x[n] W_N^{kn} = W_3^0 + W_3^k + W_3^{2k} \\ &= e^{-\frac{j2\pi k}{3}} \left[1 + 2\cos\left(\frac{2\pi k}{3}\right) \right] \\ &= \begin{cases} 3, & k = 0 \\ 0, & k = 1, 2 \end{cases} \end{aligned}$$

Together with $X[k]$ whose index is outside the interval of $0 \leq k \leq 2$, we finally have:

$$X[k] = \begin{cases} 3, & k = 0 \\ 0, & \text{otherwise} \end{cases}$$

If the length of $x[n]$ is considered as $N = 5$ such that $x[3] = x[4] = 0$, then we obtain:

$$\begin{aligned} X[k] &= \sum_{n=0}^{N-1} x[n] W_N^{kn} = W_5^0 + W_5^k + W_5^{2k} \\ &= \begin{cases} e^{-\frac{j2\pi k}{5}} \left[1 + 2\cos\left(\frac{2\pi k}{5}\right)\right], & k = 0, 1, \cdots, 4 \\ 0, & \text{otherwise} \end{cases} \end{aligned}$$

The MATLAB command for DFT computation is `fft`. The MATLAB code to produce magnitudes and phases of $X[k]$ is:

```
N=5;
x=[1 1 1 0 0];                  %append 2 zeros
subplot(2,1,1);
stem([0:N-1],abs(fft(x))); %plot magnitude response
title('Magnitude Response');
subplot(2,1,2);
stem([0:N-1],angle(fft(x)));%plot phase response
title('Phase Response');
```

According to Example 7.2 and the relationship between DFT and DFS, the DFT will approach the DTFT when we append infinite zeros at the end of $x[n]$

The MATLAB program is provided as `ex7_5.m`.

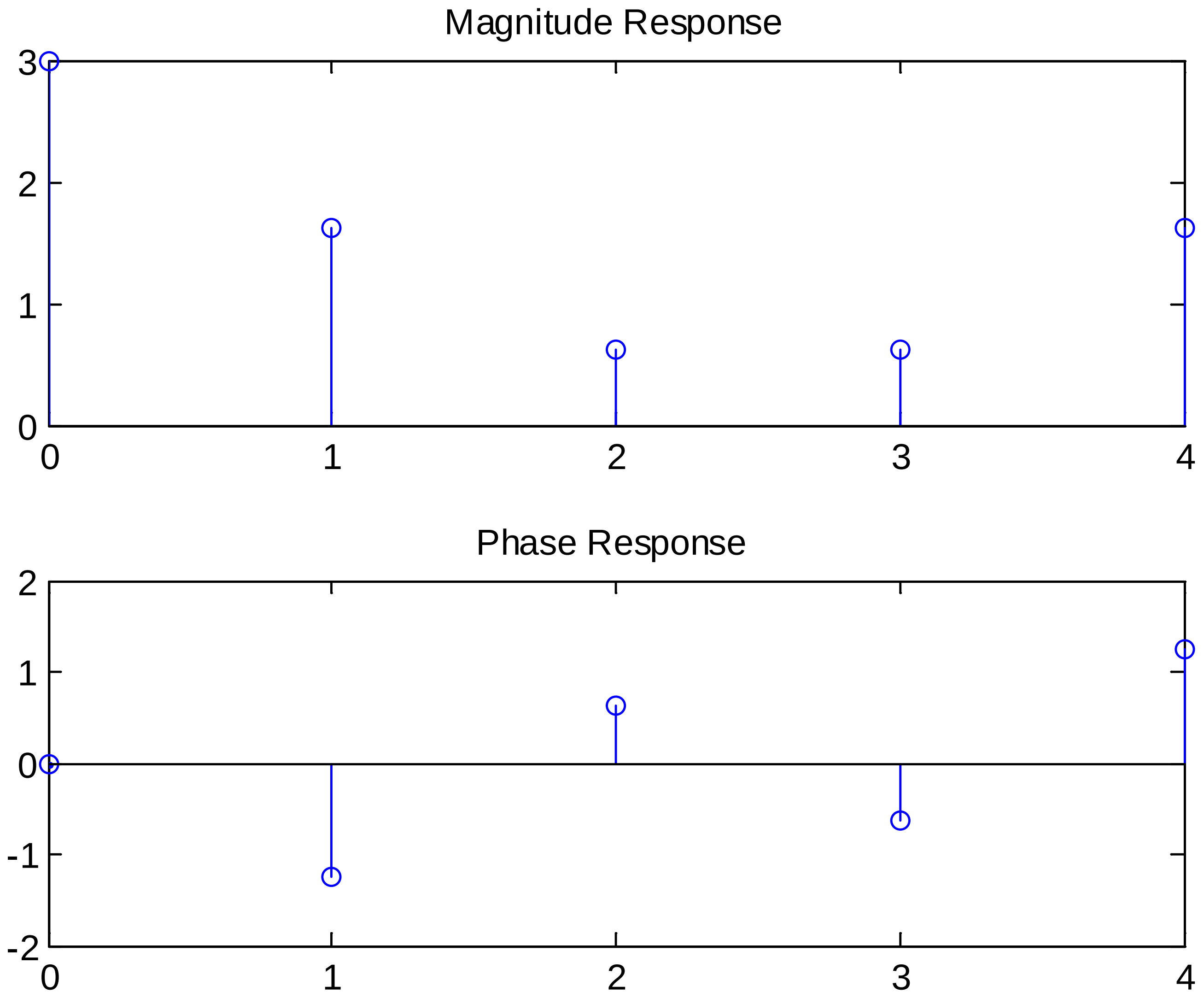


Fig.7.8: DFT plots with $N = 5$

Example 7.6

Given a discrete-time finite-duration sinusoid:

$$x[n] = 2\cos(0.7\pi n + 1), \quad n = 0, 1, \cdots, 20$$

Estimate the tone frequency using DFT.

Consider the continuous-time case first. According to (2.16), Fourier transform pair for a complex tone of frequency Ω_0 is:

$$e^{j\Omega_0 t} \leftrightarrow 2\pi\delta(\Omega - \Omega_0)$$

That is, Ω_0 can be found by locating the peak of the Fourier transform. Moreover, a real-valued tone $\cos(\Omega_0 t)$ is:

$$\cos(\Omega_0 t) = \frac{e^{j\Omega_0 t} + e^{-j\Omega_0 t}}{2}$$

From the Fourier transform of $\cos(\Omega_0 t)$, Ω_0 and $-\Omega_0$ are located from the two impulses.

Analogously, there will be two peaks which correspond to frequencies 0.7π and -0.7π in the DFT for $x[n]$.

The MATLAB code is

```
N=21;                  %number of samples is 21
A=2;                   %tone amplitude is 2
w=0.7*pi;              %frequency is 0.7*pi
p=1;                   %phase is 1
n=0:N-1;               %define a vector of size N
x=A*cos(w*n+p);        %generate tone
X=fft(x);              %compute DFT
subplot(2,1,1);
stem(n,abs(X));       %plot magnitude response
subplot(2,1,2);
stem(n,angle(X));     %plot phase response
```

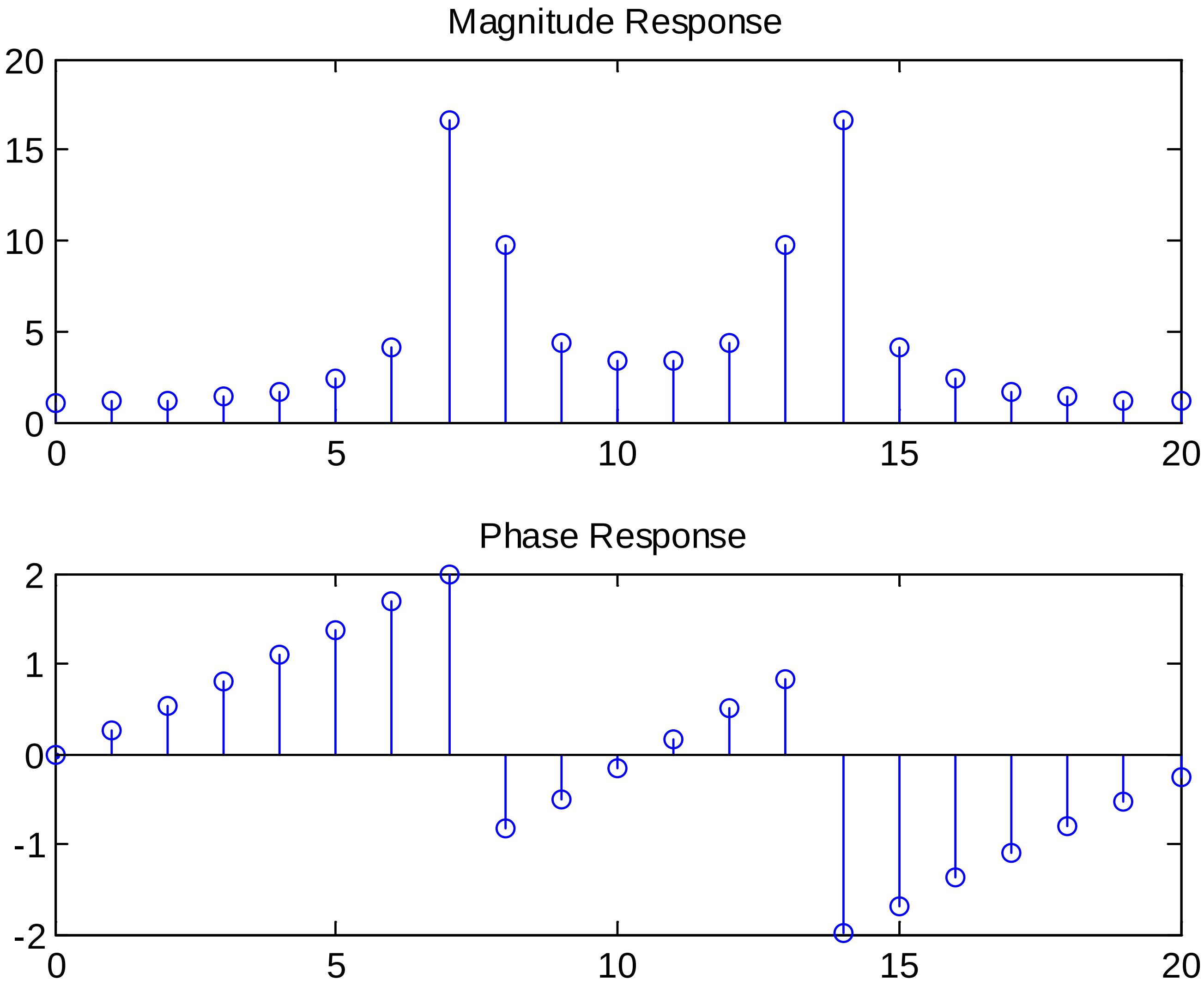


Fig.7.9: DFT plots for a real tone

```
X =
 1.0806               1.0674+0.2939i   1.0243+0.6130i
 0.9382+0.9931i   0.7756+1.5027i   0.4409+2.3159i
-0.4524+4.1068i  -6.7461+15.1792i  6.5451-7.2043i
 3.8608-2.1316i   3.3521-0.5718i   3.3521+0.5718i
 3.8608+2.1316i   6.5451+7.2043i  -6.7461-15.1792i
-0.4524-4.1068i   0.4409-2.3159i   0.7756-1.5027i
 0.9382-0.9931i   1.0243-0.6130i   1.0674-0.2939i
```

Interestingly, we observe that $\Re\{X[k]\} = \Re\{X[N-k]\}$ and $\Im\{X[k]\} = -\Im\{X[N-k]\}$. In fact, all real-valued sequences possess these properties so that we only have to compute around half of the DFT coefficients.

As the DFT coefficients are complex-valued, we search the frequency according to the magnitude plot.

There are two peaks, one at $k=7$ and the other at $k=14$ which correspond to $\omega=0.7\pi$ and $\omega=-0.7\pi$, respectively.

From Example 7.2, it is clear that the index k refers to $\omega=2\pi k/N$. As a result, an estimate of ω_0 is:

$$\hat{\omega}_0=\frac{2\pi\cdot 7}{21}\approx 0.6667\pi$$

To improve the accuracy, we append a large number of zeros to $x[n]$. The MATLAB code for $x[n]$ is now modified as:

```
x=[A*cos(w.*n+p) zeros(1,1980)];
```

where 1980 zeros are appended.

The MATLAB code is provided as `ex7_6.m` and `ex7_6_2.m`.

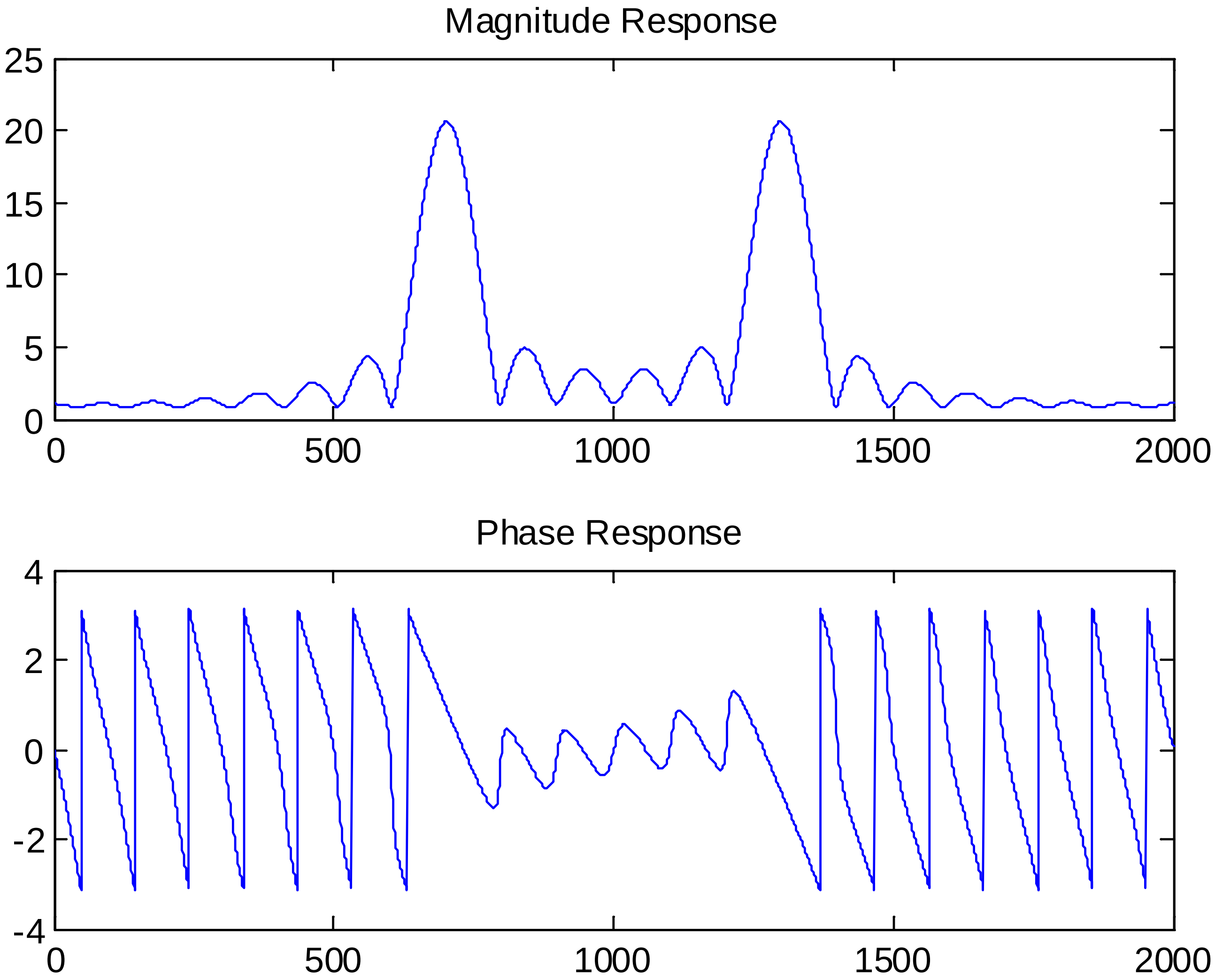


Fig.7.10: DFT plots for a real tone with zero padding

The peak index is found to be $k = 702$ with $N = 2001$. Thus

$$\hat{\omega}_0 = \frac{2\pi \cdot 702}{2001} \approx 0.7016\pi$$

Example 7.7

Find the inverse DFT coefficients for $X[k]$ which has a length of $N = 5$ and has the form of

$$X[k] = \begin{cases} 1, \ n = 0, 1, 2 \\ 0, \ n = 3, 4 \end{cases}$$

Plot $x[n]$.

Using (7.33) and Example 7.5, we have:

$$x[n] = \frac{1}{N}\sum_{n=0}^{N-1} X[k]W_N^{-kn} = \frac{1}{5}\left(W_5^0 + W_5^{-n} + W_5^{-2n}\right)$$

$$= \begin{cases} \frac{1}{5}e^{\frac{j2\pi n}{5}}\left[1 + 2\cos\left(\frac{2\pi n}{5}\right)\right], & n = 0, 1, \cdots, 4 \\ 0, & \text{otherwise} \end{cases}$$

The main MATLAB code is:

```
N=5;
X=[1 1 1 0 0];
subplot(2,1,1);
stem([0:N-1],abs(ifft(X)));
subplot(2,1,2);
stem([0:N-1],angle(ifft(X)));
```

The MATLAB program is provided as `ex7_7.m`.

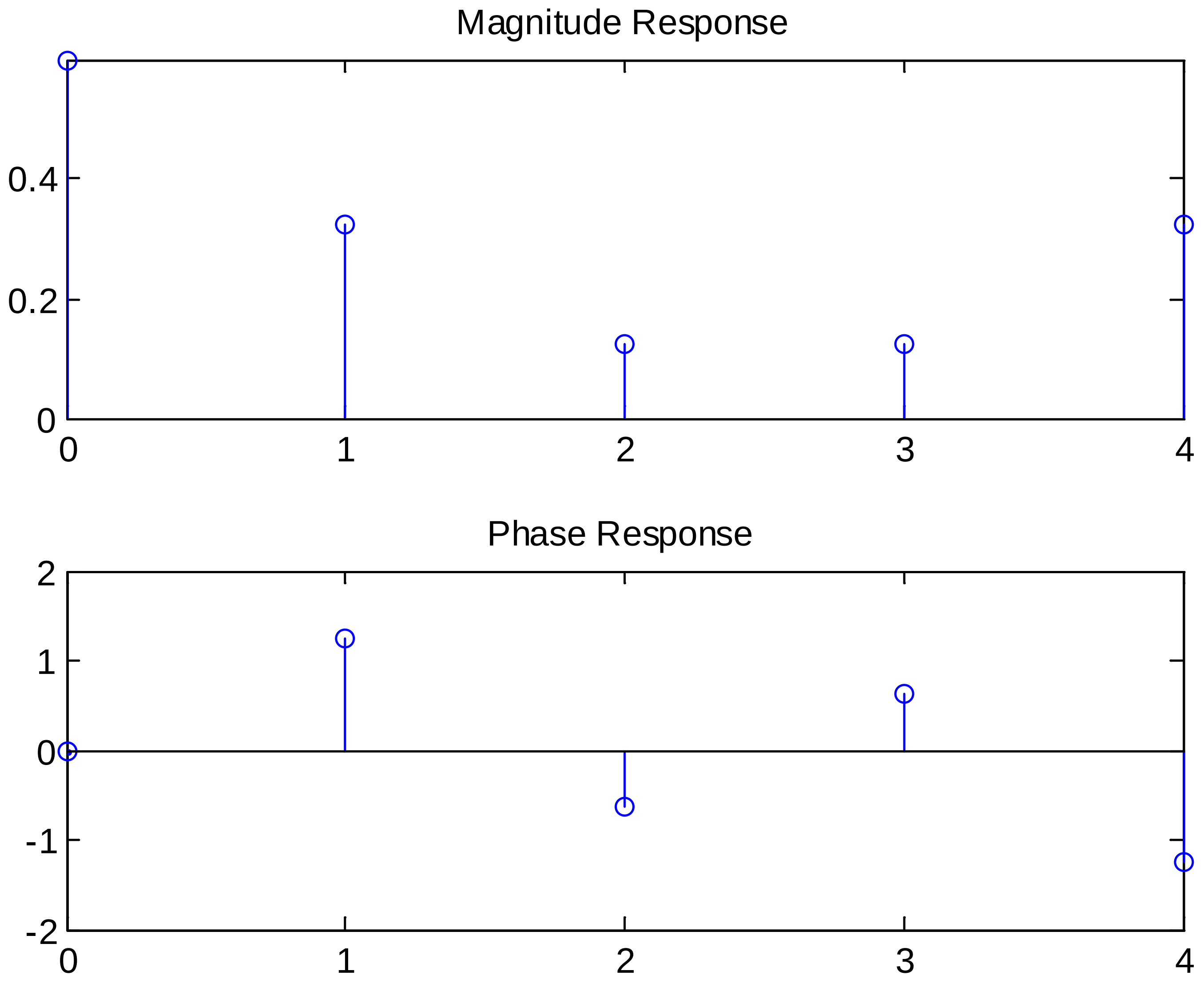


Fig.7.11: Inverse DFT plots

Properties of DFT

Since DFT pair is equal to DFS pair within $[0, N-1]$, their properties will be identical if we take care of the values of $x[n]$ and $X[k]$ when the indices are outside the interval

1. Linearity

Let $(x_1[n], X_1[k])$ and $(x_2[n], X_2[k])$ be two DFT pairs with the same duration of N. We have:

$$ax_1[n] + bx_2[n] \leftrightarrow aX_1[k] + bX_2[k] \qquad (7.34)$$

Note that if $x_1[n]$ and $x_2[n]$ are of different lengths, we can properly append zero(s) to the shorter sequence to make them with the same duration.

2. Circular Shift of Sequence

If $x[n] \leftrightarrow X[k]$, then

$$x[(n-m) \mod (N)] \leftrightarrow W_N^{km} X[k] \quad (7.35)$$

Note that in order to make sure that the resultant time index is within the interval of $[0, N-1]$, we need circular shift, which is defined as

$$(n-m) \mod (N) = n - m + r \cdot N \quad (7.36)$$

where the integer r is chosen such that

$$0 \leq n - m + r \cdot N \leq N - 1 \quad (7.37)$$

Example 7.8

Determine $x_1[n] = x[(n-2) \mod (4)]$ where $x[n]$ is of length 4 and has the form of:

$$x[n] = \begin{cases} 1, & n = 0 \\ 3, & n = 1 \\ 2, & n = 2 \\ 4, & n = 3 \end{cases}$$

According to (7.36)-(7.37) with $N = 4$, $x_1[n]$ is determined as:

$$\begin{aligned} x_1[0] &= x[(0-2) \mod (4)] = x[2] = 2, \quad r = 1 \\ x_1[1] &= x[(1-2) \mod (4)] = x[3] = 4, \quad r = 1 \\ x_1[2] &= x[(2-2) \mod (4)] = x[0] = 1, \quad r = 0 \\ x_1[3] &= x[(3-2) \mod (4)] = x[1] = 3, \quad r = 0 \end{aligned}$$

3. Duality

If $x[n] \leftrightarrow X[k]$, then

$$X[n] \leftrightarrow Nx[(-k) \mod (N)] \tag{7.38}$$

4. Symmetry

If $x[n] \leftrightarrow X[k]$, then

$$x^*[n] \leftrightarrow X^*[(-k) \mod (N)] \tag{7.39}$$

and

$$x^*[(-n) \mod (N)] \leftrightarrow X^*[k] \tag{7.40}$$

5. Circular Convolution

Let $(x_1[n], X_1[k])$ and $(x_2[n], X_2[k])$ be two DFT pairs with the same duration of N. We have

$$x_1[n] \otimes_N x_2[n] = \sum_{m=0}^{N-1} x_1[m] x_2[(n-m) \mod (N)] \leftrightarrow X_1[k] X_2[k] \quad (7.41)$$

where $\otimes_N$ is the circular convolution operator.

Fast Fourier Transform

FFT is a fast algorithm for DFT and inverse DFT computation.

Recall (7.32):

$$X[k]=\sum_{n=0}^{N-1} x[n]W_N^{kn}, \quad 0\le k\le N-1 \tag{7.42}$$

Each $X[k]$ involves N and $(N-1)$ complex multiplications and additions, respectively.

Computing all DFT coefficients requires N^2 complex multiplications and $N(N-1)$ complex additions.

Assuming that $N=2^v$, the corresponding computational requirements for FFT are $0.5N\log_2(N)$ complex multiplications and $N\log_2(N)$ complex additions.

N	Direct Computation		FFT	
	Multiplication N^2	Addition $N(N-1)$	Multiplication $0.5N\log_2(N)$	Addition $N\log_2(N)$
2	4	2	1	2
8	64	56	12	24
32	1024	922	80	160
64	4096	4022	192	384
2^{10}	1048576	1047552	5120	10240
2^{20}	$\sim 10^{12}$	$\sim 10^{12}$	$\sim 10^{7}$	$\sim 2\times 10^{7}$

Table 7.1: Complexities of direct DFT computation and FFT

Basically, FFT makes use of two ideas in its development:

- Decompose the DFT computation of a sequence into successively smaller DFTs

- Utilize two properties of $W_N^k = e^{-j2\pi k/N}$:

  - complex conjugate symmetry property:

$$W_N^{k(N-n)} = W_N^{-kn} = \left(W_N^{kn}\right)^* \qquad (7.43)$$

  - periodicity in n and k:

$$W_N^{kn} = W_N^{k(n+N)} = W_N^{n(k+N)} \qquad (7.44)$$

Decimation-in-Time Algorithm

The basic idea is to compute (7.42) according to

$$X[k] = \sum_{n=\text{even}}^{N-1} x[n] W_N^{kn} + \sum_{n=\text{odd}}^{N-1} x[n] W_N^{kn} \tag{7.45}$$

Substituting $n = 2r$ and $n = 2r + 1$ for the first and second summation terms:

$$\begin{aligned} X[k] &= \sum_{r=0}^{N/2-1} x[2r] W_N^{2rk} + \sum_{r=0}^{N/2-1} x[2r+1] W_N^{(2r+1)k} \\ &= \sum_{r=0}^{N/2-1} x[2r] \left(W_N^2\right)^{rk} + W_N^k \sum_{r=0}^{N/2-1} x[2r+1] \left(W_N^2\right)^{rk} \end{aligned} \tag{7.46}$$

Using $W_N^2 = W_{N/2}$ since $W_N^2 = e^{-j2\pi/N \cdot 2} = e^{-j2\pi/(N/2)}$, we have:

$$\begin{aligned} X[k] &= \sum_{r=0}^{N/2-1} x[2r] W_{N/2}^{rk} + W_N^k \sum_{r=0}^{N/2-1} x[2r+1] W_{N/2}^{rk} \\ &= G[k] + W_N^k \cdot H[k], \quad k = 0, 1, \cdots, N-1 \end{aligned} \tag{7.47}$$

where $G[k]$ and $H[k]$ are the DFTs of the even-index and odd-index elements of $x[n]$, respectively. That is, $X[k]$ can be constructed from two $N/2$-point DFTs, namely, $G[k]$ and $H[k]$.

Further simplifications can be achieved by writing the N equations as 2 groups of $N/2$ equations as follows:

$$X[k] = G[k] + W_N^k \cdot H[k], \quad k = 0, 1, \cdots, N/2-1 \tag{7.48}$$

and

$$\begin{aligned} X[k+N/2] &= \sum_{r=0}^{N/2-1} x[2r]W_{N/2}^{r(k+N/2)} + W_N^{k+N/2} \sum_{r=0}^{N/2-1} x[2r+1]W_{N/2}^{r(k+N/2)} \\ &= \sum_{r=0}^{N/2-1} x[2r]W_{N/2}^{rk} - W_N^{k} \sum_{r=0}^{N/2-1} x[2r+1]W_{N/2}^{rk} \\ &= G[k] - W_N^k \cdot H[k], \quad k = 0, 1, \cdots, N/2-1 \end{aligned} \tag{7.49}$$

with the use of $W_{N/2}^{N/2} = 1$ and $W_N^{N/2} = -1$. Equations (7.48) and (7.49) are known as the butterfly merging equations.

Noting that $N/2$ multiplications are also needed to calculate $W_N^k H[k]$, the number of multiplications is reduced from N^2 to $2(N/2)^2 + N/2 = N(N+1)/2$.

The decomposition step of (7.48)-(7.49) is repeated v times until 1-point DFT is reached.

Decimation-in-Frequency Algorithm

The basic idea is to decompose the frequency-domain sequence $X[k]$ into successively smaller subsequences.

Recall (7.42) and employing $W_N^{2r(n+N/2)} = W_N^{2nr} \cdot W_N^{rN} = W_N^{2nr}$ and $W_N^2 = W_{N/2}$, the even-index DFT coefficients are:

$$X[2r] = \sum_{n=0}^{N-1} x[n] W_N^{n(2r)} = \sum_{n=0}^{N/2-1} x[n] W_N^{2nr} + \sum_{n=N/2}^{N-1} x[n] W_N^{2nr}$$
$$= \sum_{n=0}^{N/2-1} x[n] W_N^{2nr} + \sum_{n=0}^{N/2-1} x[n+N/2] W_N^{2r(n+N/2)}$$
$$= \sum_{n=0}^{N/2-1} (x[n] + x[n+N/2]) \cdot W_{N/2}^{nr}, \quad r = 0, 1, \cdots, N/2-1 \quad (7.50)$$

Using $W_N^{Nr}=1$ and $W_N^{N/2}=-1$, the odd-index coefficients are:

$$\begin{aligned}X[2r+1] &= \sum_{n=0}^{N/2-1} x[n]W_N^{n(2r+1)} + \sum_{n=N/2}^{N-1} x[n]W_N^{n(2r+1)} \\ &= \sum_{n=0}^{N/2-1} x[n]W_N^n W_{N/2}^{nr} + \sum_{n=0}^{N/2-1} x[n+N/2]W_N^{(n+N/2)(2r+1)} \\ &= \sum_{n=0}^{N/2-1} x[n]W_N^n W_{N/2}^{nr} + W_N^{N/2(2r+1)} \sum_{n=0}^{N/2-1} x[n+N/2]W_N^{n(2r+1)} \\ &= \sum_{n=0}^{N/2-1} \left(x[n]-x[n+N/2]\right) W_N^n \cdot W_{N/2}^{nr}, \quad r=0,1,\cdots,N/2-1 \end{aligned} \quad (7.51)$$

$X[2r]$ and $X[2r+1]$ are equal to $N/2$-point DFTs of $(x[n]+x[n+N/2])$ and $(x[n]-x[n+N/2])\,W_N^n$, respectively. The decomposition step of (7.50)-(7.51) is repeated v times until 1-point DFT is reached

Fast Convolution with FFT

The convolution of two finite-duration sequences

$$y[n] = x_1[n] \otimes x_2[n]$$

where $x_1[n]$ is of length N_1 and $x_2[n]$ is of length N_2 requires computation of ($N_1 + N_2$-1) samples which corresponds to $N_1 N_2 - \min\{N_1, N_2\}$ complex multiplications

An alternate approach is to use FFT:

$$y[n] = \mathrm{IFFT}\{\mathrm{FFT}\{x_1[n]\} \times \mathrm{FFT}\{x_2[n]\}\}$$

In practice:

- Choose the minimum $N \geq N_1 + N_2 - 1$ and is power of 2
- Zero-pad $x_1[n]$ and $x_2[n]$ to length N, say, $\breve{x}_1[n]$ and $\breve{x}_2[n]$
- $\breve{y}[n] = \mathrm{IFFT}\{\mathrm{FFT}\{\breve{x}_1[n]\} \times \mathrm{FFT}\{\breve{x}_2[n]\}\}$

From (7.33), the inverse DFT has a factor of $1/N$, the IFFT thus requires $N+(N/2)\log_2(N)$ multiplications. As a result, the total multiplications for $\breve{y}[n]$ is $2N+(3N/2)\log_2(N)$

Using FFT is more computationally efficient than direct convolution computation for longer data lengths:

N_1	N_2	N	$N_1N_2-\min\{N_1,N_2\}$	$2N+(3N/2)\log_2(N)$
2	5	8	8	52
10	15	32	140	304
50	80	256	3950	3584
50	1000	2048	49950	37888
512	10000	16384	4119488	376832

MATLAB and C source codes for FFT can be found at:

http://www.ece.rutgers.edu/~orfanidi/intro2sp/#progs