Lecture 4. Analog Communications
Part II. Frequency Modulation (FM)

- Angle Modulation (FM and PM)
- Spectral Characteristics of FM signals
- FM Modulator and Demodulator

More About Amplitude Modulation

- Simple and bandwidth efficient (compared to FM)

- High requirement on amplifiers
  - Linear amplifiers are difficult to achieve in applications.
- Low fidelity performance
  - Noise enhancement in quiet periods
  - No tradeoff between bandwidth and fidelity performance

Angle Modulation

An angle-modulated signal can be written as

$$s_{AnM}(t) = A\cos(\Psi(t)) \qquad \Psi(t)\text{: Instantaneous Phase}$$

$$f(t) = \frac{1}{2\pi}\cdot\frac{d\Psi(t)}{dt} \qquad f(t)\text{: Instantaneous Frequency}$$

A typical carrier signal: $\Psi(t) = 2\pi f_c t + \theta(t), \quad f(t) = f_c + \frac{1}{2\pi}\cdot\frac{d\theta(t)}{dt}$

- Phase Modulation (PM):

$$\theta(t) = \alpha s(t) \;\Rightarrow\; s_{PM}(t) = A\cos(2\pi f_c t + \alpha s(t))$$

- Frequency Modulation (FM):

$$\frac{1}{2\pi}\cdot\frac{d\theta(t)}{dt} = ks(t) \;\Rightarrow\; s_{FM}(t) = A\cos(2\pi(f_c t + k\int_{-\infty}^{t} s(\tau)d\tau))$$

Phase Deviation and Frequency Deviation

Phase Modulation (PM) signal $s_{PM}(t) = A\cos(2\pi f_c t + \alpha s(t))$

Instantaneous Phase: $\Psi(t) = 2\pi f_c t + \alpha s(t)$

Phase Deviation: $|\alpha s(t)|$

Maximum (Peak) Phase Deviation: $\max_t |\alpha s(t)|$

> Peak phase deviation represents the maximum phase difference between the transmitted signal and the carrier signal.

Frequency Modulation (FM) signal $s_{FM}(t) = A\cos(2\pi(f_c t + k\int_{-\infty}^{t} s(\tau)d\tau))$

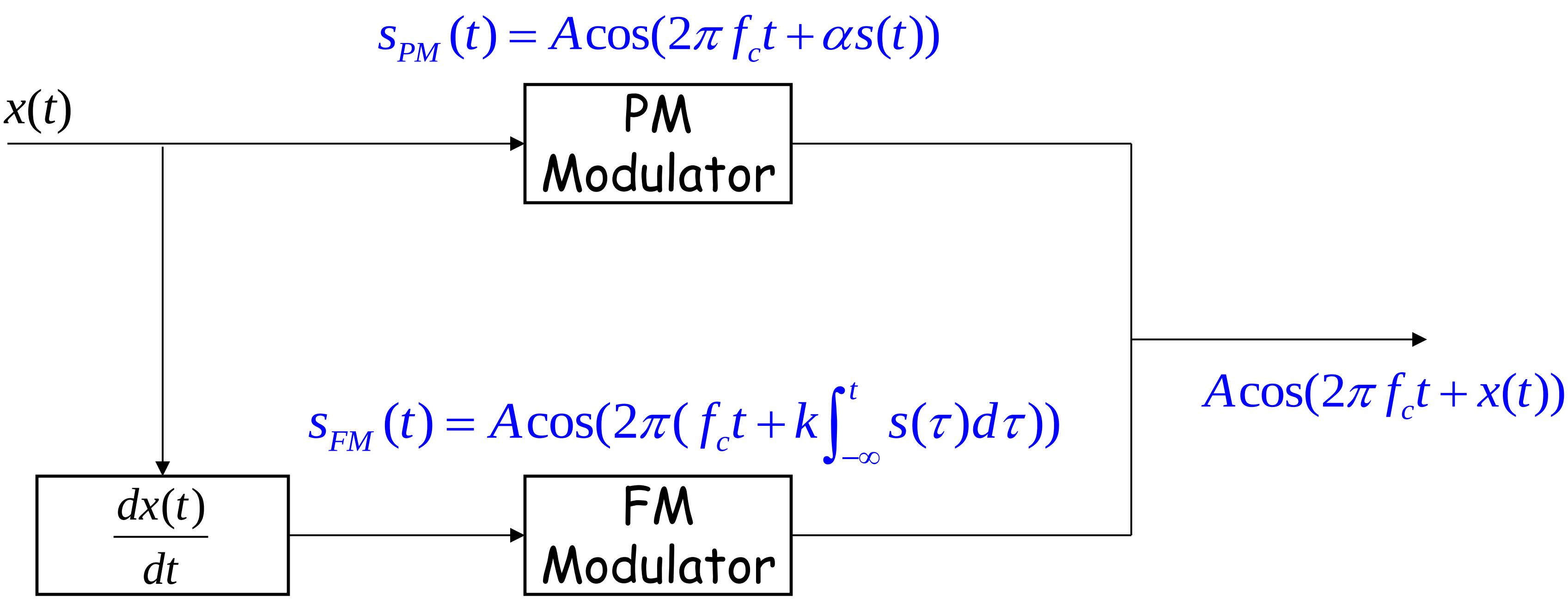
Instantaneous Frequency: $f(t) = \frac{1}{2\pi} \cdot \frac{d\Psi(t)}{dt} = f_c + ks(t) \text{ Hz}$

Frequency Deviation: $|ks(t)|$

Maximum (Peak) Frequency Deviation: $\max_t |ks(t)|$

> Peak frequency deviation represents the maximum departure of the instantaneous frequency from the carrier frequency.

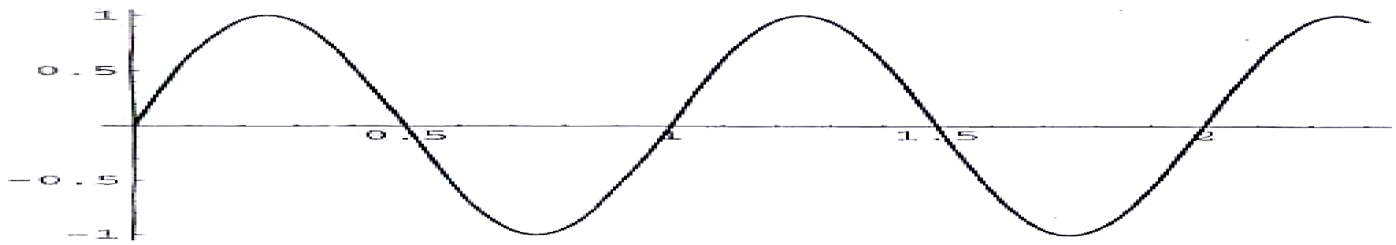
Relationship between PM and FM



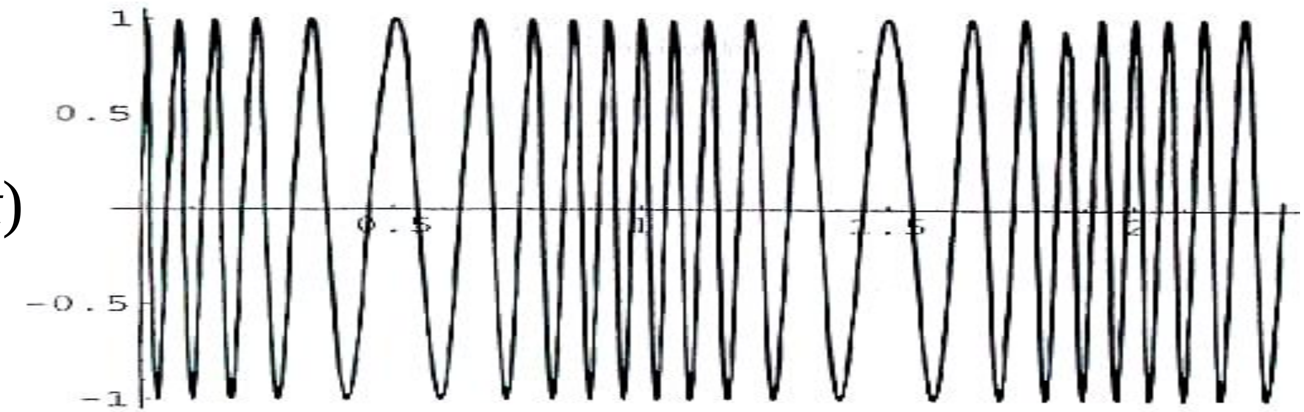
- Phase modulation of the carrier with a message signal is equivalent to frequency modulation of the carrier with the derivative of the message signal.
- We will only focus on FM in the following.

FM Signal

$s(t)$



$s_{FM}(t)$



FM Single Tone Modulated Waveform

Spectral Characteristics of Frequency Modulated Signals

A False Start

- Frequency Modulation (FM):

$$s_{FM}(t) = A\cos(2\pi(f_c t + k\int_{-\infty}^{t} s(\tau)d\tau))$$

Instantaneous Frequency $f(t) = f_c + ks(t)$

Suppose that the peak amplitude of s(t) is m_s. Then the maximum and minimum values of the instantaneous frequency of the modulated signal would be f_c+km_s and f_c-km_s. Then the spectral components of the FM signal would be within the frequency band of $[f_c-km_s, f_c+km_s]$ with the bandwidth of $2km_s$.

$\Rightarrow$ The bandwidth of the FM signal could be arbitrarily small by using an arbitrarily small k!

Where is the fallacy in this reasoning?

Too good to be true ☹

FM Sinusoidal Signal

- Let us first assume the message signal $s(t)$ is a sinusoidal signal:

$$s(t) = A_m \cos(2\pi f_m t)$$

$$s_{FM}(t) = A\cos\{2\pi[f_c t + k\int_{-\infty}^{t} A_m \cos(2\pi f_m \tau)d\tau]\}$$

$$= A\cos\{2\pi[f_c t + \frac{kA_m}{2\pi f_m}\sin(2\pi f_m t)]\}$$

$$= A\cos[2\pi f_c t + \frac{\Delta f}{f_m}\sin(2\pi f_m t)]$$

$$= A\cos[2\pi f_c t + \beta \sin(2\pi f_m t)]$$

Instantaneous phase

Instantaneous frequency:

$$f(t) = f_c + kA_m \cos(2\pi f_m t)$$

Peak frequency deviation:

$$\Delta f = \max|ks(t)| = kA_m$$

- Peak frequency deviation $\Delta f = kA_m$ is proportional to A_m, the amplitude of the message signal s(t).
- $\beta = \frac{\Delta f}{f_m}$ is defined as the modulation index.

Modulation Indices of AM-DSB-C and FM

AM-DSB-C -- with a message signal $s(t)$: $m=\frac{\max[s(t)+c]-\min[s(t)+c]}{\max[s(t)+c]+\min[s(t)+c]}$

With a sinusoidal message signal $s(t)=A_m\cos(2\pi f_m t)$:

$$s_{AM-DSB-C}(t)=A(s(t)+c)\cos(2\pi f_c t)$$

$$=Ac(m\cos(2\pi f_m t)+1)\cos(2\pi f_c t)$$

$$m=\frac{(A_m+c)-(-A_m+c)}{(A_m+c)+(-A_m+c)}=\frac{A_m}{c}$$

FM -- With a sinusoidal message signal $s(t)=A_m\cos(2\pi f_m t)$:

$$s_{FM}(t)=A\cos[2\pi f_c t+\beta\sin(2\pi f_m t)]$$

$$\beta=\frac{kA_m}{f_m}$$

With a general message signal $s(t)$: $\beta=\frac{k\max|s(t)|}{B_s}$

FM Sinusoidal Signal (Cont'd)

$s_{FM}(t)$ can be expanded as an infinite Fourier series:

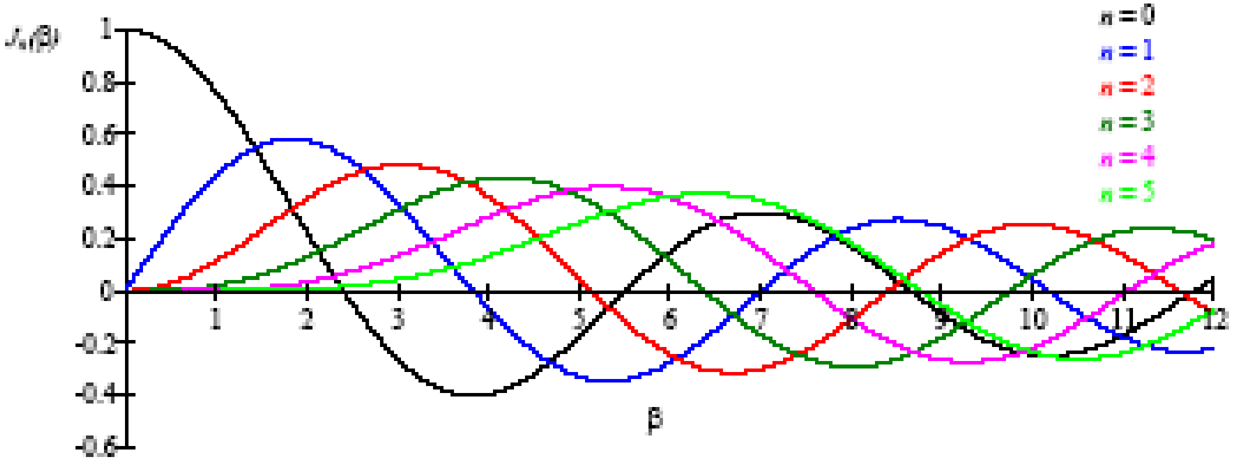
$$s_{FM}(t) = A\cos[2\pi f_c t + \beta\sin(2\pi f_m t)]$$
$$= A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c + nf_m)t]$$

$J_n(\beta)$ is called the Bessel Function of the first kind and of order n, which is defined by

$$J_n(\beta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{j(\beta\sin x - nx)}dx$$

Read Reference [1] (Sec. 3.3.2) to see the derivation of $s_{FM}(t)$ and more details about Bessel Function.

A Little Bit about Bessel Function



For small values of β: $$J_n(\beta) \approx \frac{\beta^n}{2^n n!}$$

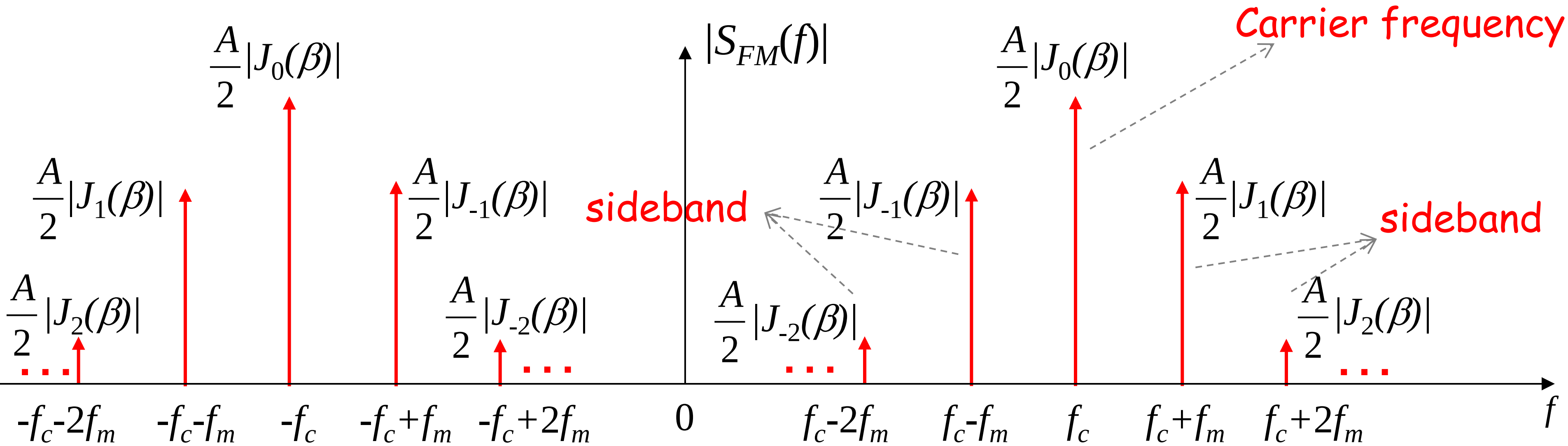
For large values of β: $$J_n(\beta) \approx \sqrt{\frac{2}{\pi\beta}} \cos\left(\beta - \frac{\pi}{4} - \frac{n\pi}{2}\right)$$

Symmetry property: $$J_{-n}(\beta) = \begin{cases} J_n(\beta) & n \text{ even} \\ -J_n(\beta) & n \text{ odd} \end{cases}$$ $$|J_{-n}(\beta)| = |J_n(\beta)|$$

Magnitude Spectrum of FM Sinusoidal Signals

$$s_{FM}(t) = A\sum_{n=-\infty}^{\infty} J_n(\beta)\cos[2\pi(f_c+nf_m)t] = \frac{A}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)e^{j2\pi(f_c+nf_m)t} + \frac{A}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)e^{-j2\pi(f_c+nf_m)t}$$

$$S_{FM}(f) = \frac{A}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f-f_c-nf_m) + \frac{A}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f+f_c+nf_m)$$

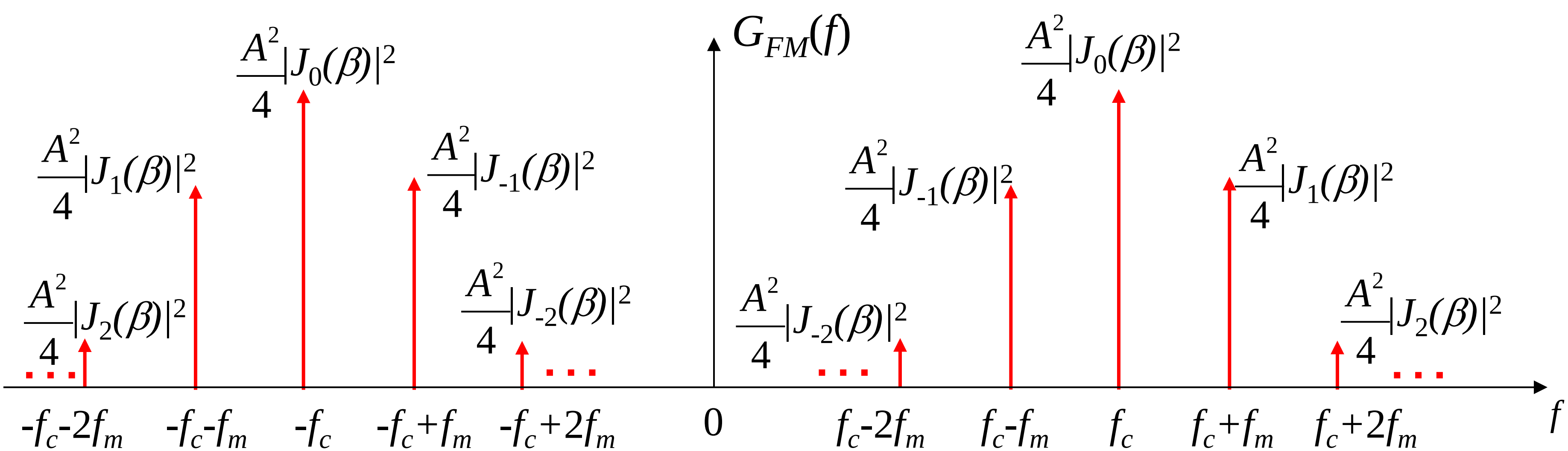


The bandwidth of an FM sinusoidal signal is ∞.

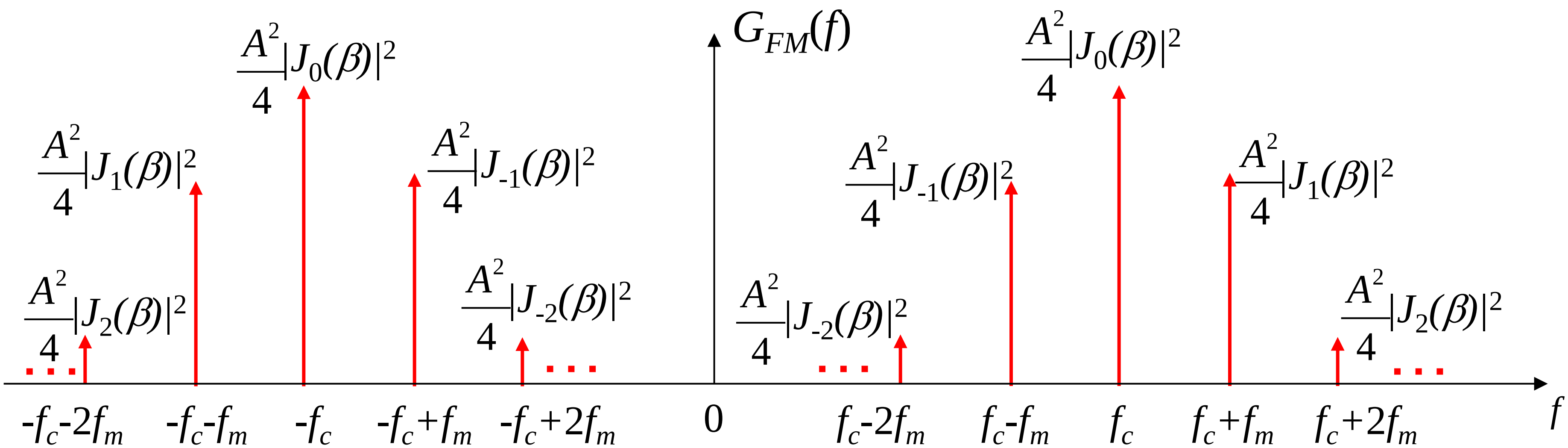
Power Spectrum of FM Sinusoidal Signals

$$S_{FM}(f) = \frac{A}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f - f_c - nf_m) + \frac{A}{2}\sum_{n=-\infty}^{\infty} J_n(\beta)\delta(f + f_c + nf_m)$$

$$G_{FM}(f) = \frac{A^2}{4}\sum_{n=-\infty}^{\infty} |J_n(\beta)|^2 \delta(f - f_c - nf_m) + \frac{A^2}{4}\sum_{n=-\infty}^{\infty} |J_n(\beta)|^2 \delta(f + f_c + nf_m)$$



Power Spectrum of FM Sinusoidal Signals

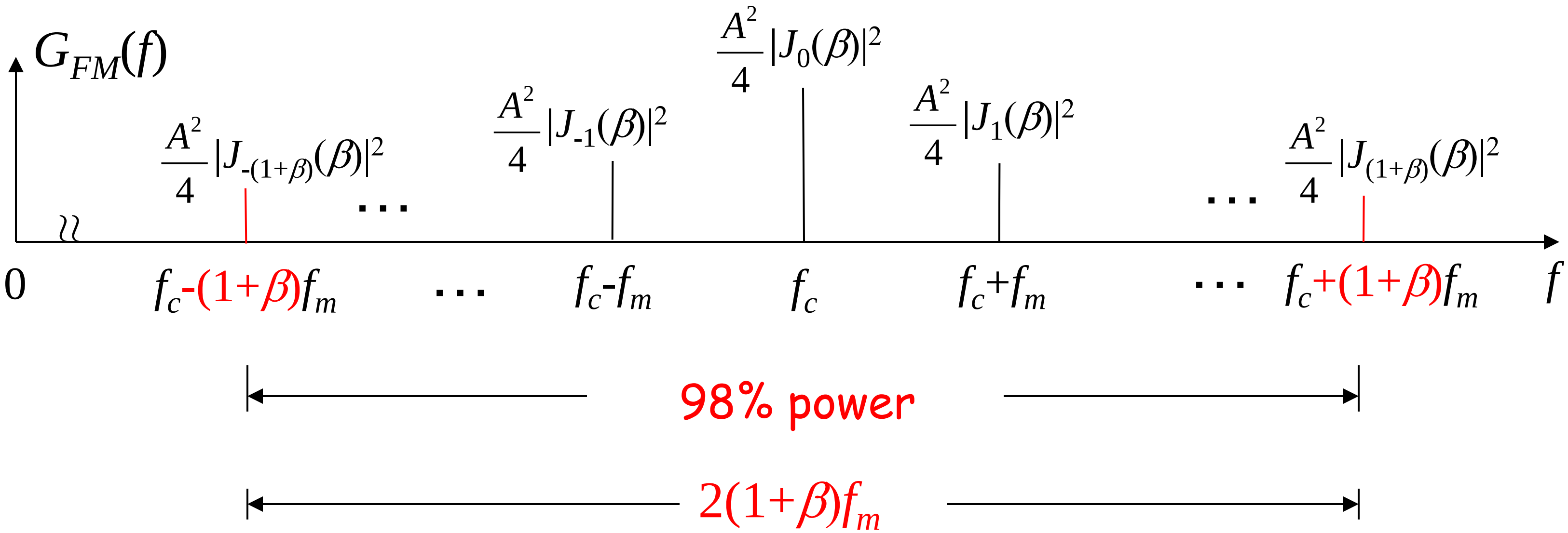


$$P_t = \underbrace{\frac{A^2}{2}\cdot J_0^2(\beta)}_{\text{Power at carrier}} + \underbrace{\frac{A^2}{2}\cdot\left(\sum_{n=1}^{\infty} J_n^2(\beta) + \sum_{n=-\infty}^{-1} J_n^2(\beta)\right)}_{\text{Power at sidebands}} = \frac{A^2}{2}$$

$$\Rightarrow \boxed{J_0^2(\beta) + 2\sum_{n=1}^{\infty} J_n^2(\beta) = 1}$$

$$s_{FM}(t) = A\cos[2\pi f_c t + \beta\sin(2\pi f_m t)] \;\Rightarrow\; P_t = \frac{A^2}{2}$$

Effective Bandwidth of FM Sinusoidal Signals



Carson's Rule: The effective bandwidth of an FM sinusoidal signal is given by

$$2(1+\beta)f_m = 2(f_m + \Delta f)$$

β<<1: narrowband FM
Large β: wideband FM

Bandwidth Efficiency of FM Signals

- Bandwidth Efficiency of FM sinusoidal signals:

$$\gamma_{FM} = \frac{f_m}{2(1+\beta) f_m} = \frac{1}{2(1+\beta)} < 50\%$$

✓ Worse than AM systems.

- Bandwidth Efficiency of FM signals:

$$\gamma_{FM} = \frac{B_s}{2(1+\beta) B_s} = \frac{1}{2(1+\beta)} < 50\%$$

Edwin Howard Armstrong: Inventor of Modern FM Radio

(December 18, 1890 – January 31, 1954)

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- Bandwidth Efficiency of FM signals:

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✓ Worse than AM systems.

✓ Larger β:

- Lower bandwidth efficiency
- Better fidelity performance

FM systems can provide much better fidelity performance than AM systems by sacrificing the bandwidth efficiency.

Pros and Cons of FM

- Less requirement on amplifiers (constant amplitude)
- Flexible tradeoff between channel bandwidth and fidelity performance

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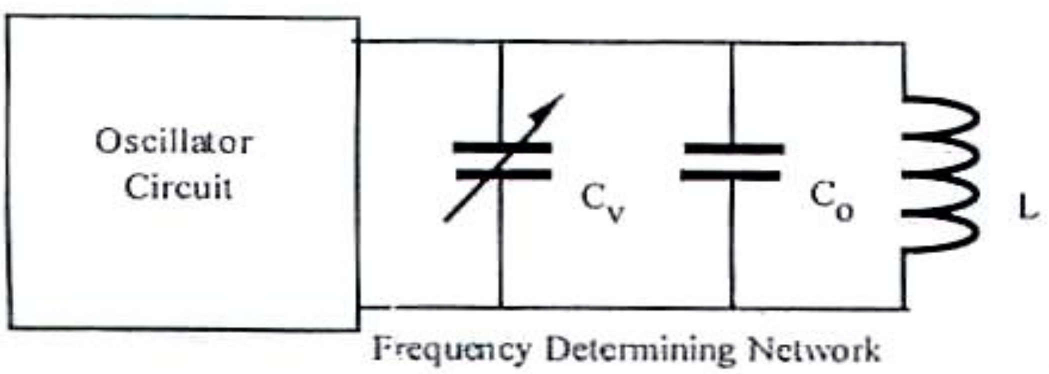
- Low bandwidth efficiency

FM Modulator and Demodulator

Direct FM

The carrier signal used in a direct FM system can be generated by a sinusoidal oscillator circuit where the oscillator frequency is controllable.

For example, in the circuit shown below, the oscillator frequency can be adjusted by tuning the capacitance of C_v.



Resonant Frequency :

$$f_o = \frac{1}{2\pi\sqrt{L(C_o + C_v)}}$$

Indirect FM

In practice, it is very difficult to construct highly stable oscillators that can be voltage-controlled accurately. Therefore, direct FM is not commonly used in FM broadcast transmitters. It is only used in applications where low equipment cost is more important than frequency stability, e.g. radio control.

Indirect FM is more widely adopted as it is easier for practical circuit realization. An indirect FM modulator includes two steps:

- A highly stable narrowband FM (NBFM) modulator (i.e., with a small β) that does not require voltage-controlled oscillators; and
- A frequency multiplier to increase β. This is usually done together with frequency shifting and bandwidth expanding.

Narrowband FM (NBFM) Modulator

NarrowBand FM (NBFM) is a special case of FM where the modulation index, β, is small (usually $\beta << 1$). Recall that an FM signal is given by:

$$s_{FM}(t) = A\cos\{2\pi[f_c t + k\int_{-\infty}^{t} s(\tau)d\tau]\}$$
$$= A\cos(2\pi f_c t + \theta(t)) \qquad \theta(t) = 2\pi k\int_{-\infty}^{t} s(\tau)d\tau$$
$$= A\cos(2\pi f_c t)\cos(\theta(t)) - A\sin(2\pi f_c t)\sin(\theta(t))$$

If $|\theta(t)|$ is small then we have the following approximations:

$$\cos(\theta(t)) \approx 1 \quad \text{and} \quad \sin(\theta(t)) \approx \theta(t)$$

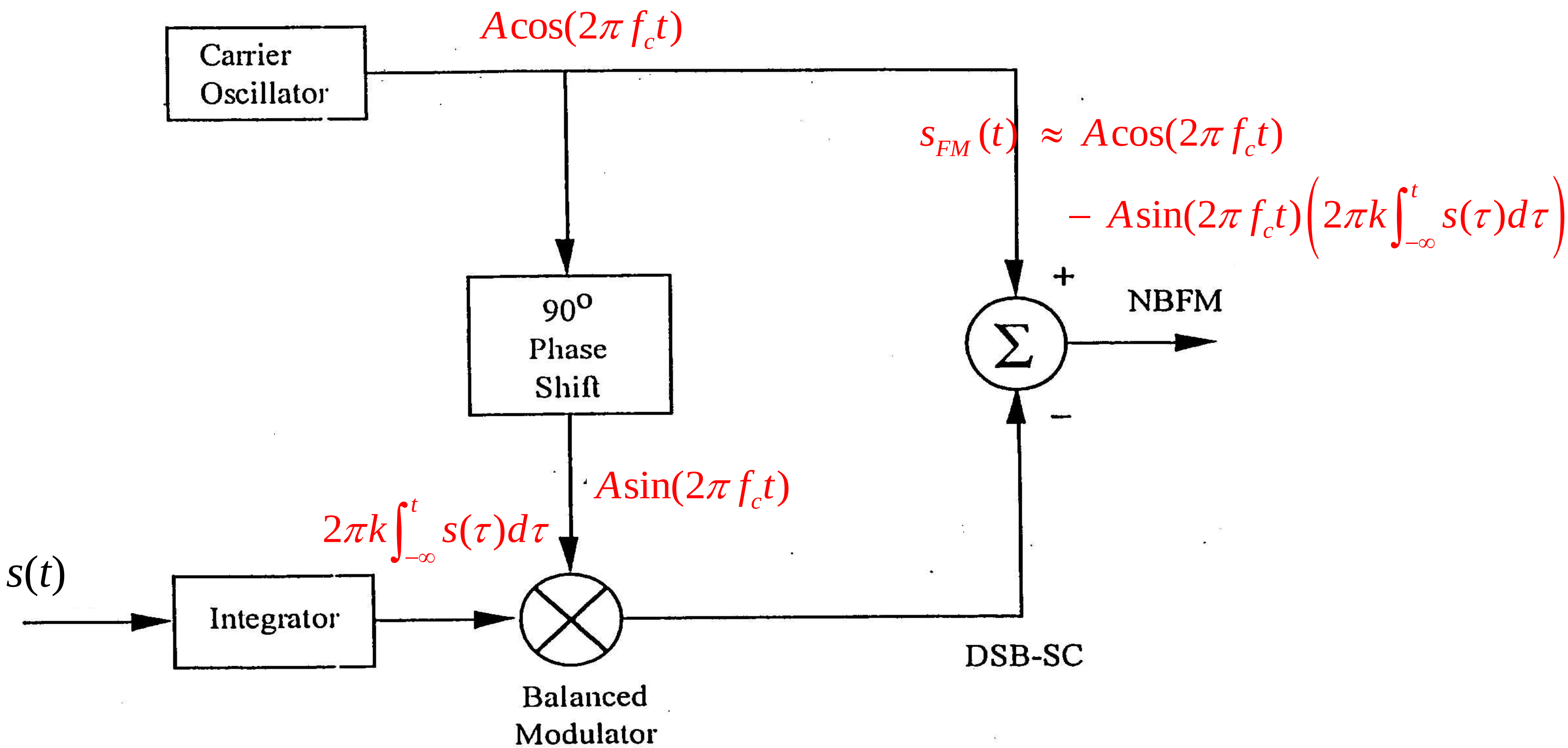
As a result,

$$s_{FM}(t) \approx A\cos(2\pi f_c t) - A\theta(t)\sin(2\pi f_c t) \qquad \theta(t) = 2\pi k\int_{-\infty}^{t} s(\tau)d\tau$$

⇩

$$s_{FM}(t) \approx A\cos(2\pi f_c t) - A\sin(2\pi f_c t)\left(2\pi k\int_{-\infty}^{t} s(\tau)d\tau\right)$$

Armstrong FM Modulator



Frequency Multiplier

The main advantage of Armstrong FM modulator is its high frequency stability. While the Armstrong modulator is only suitable for FM with a small β. For large β, a frequency multiplier can be used at the output of the Armstrong modulator.

In particular, let us consider a frequency doubler defined as:

$$e_o(t) = e_i^2(t).$$

If $e_i(t)$ is an FM signal, e.g., $e_i(t) = \cos(2\pi f_c t + \beta \sin 2\pi f_m t)$, we have

$$e_o(t) = \cos^2(2\pi f_c t + \beta \sin 2\pi f_m t) = 0.5[1+\cos(2\times 2\pi f_c t + 2\times\beta \sin 2\pi f_m t)]$$

Both β and carrier frequency have been doubled.

A frequency multiplier can be formed by cascading several doublers.

FM Demodulator: Slope Detection

Finally, we briefly discuss the FM demodulator.

Let us take the derivative of an FM signal $s_{FM}(t) = A\cos\left(2\pi\left(f_c t + k\int_{-\infty}^{t} s(\tau)d\tau\right)\right)$:

$$\frac{ds_{FM}(t)}{dt} = \left(2\pi f_c + 2\pi k s(t)\right)\cdot\left(-A\sin\left(2\pi\left(f_c t + k\int_{-\infty}^{t} s(\tau)d\tau\right)\right)\right)$$

The envelope of this signal is:

$$A\left(2\pi f_c + 2\pi k s(t)\right)$$

We can then recover $s(t)$ from this envelope signal by removing its DC component.

Summary of FM and AM

		Complexity	Bandwidth Efficiency	Fidelity (evaluated by output SNR)
AM	DSB-SC	high	50%	$\sim P_t$
	DSB-C	low		
	SSB	high	100%	(DSB-C has a lower SNR)
	VSB	high	$50\% < \frac{B_s}{B_s + \Delta} < 100\%$	
FM		moderate	$\frac{B_s}{2(1+\beta)B_s} < 50\%$	$\sim P_t$ $\sim \beta^2$

More bandwidth, better fidelity performance