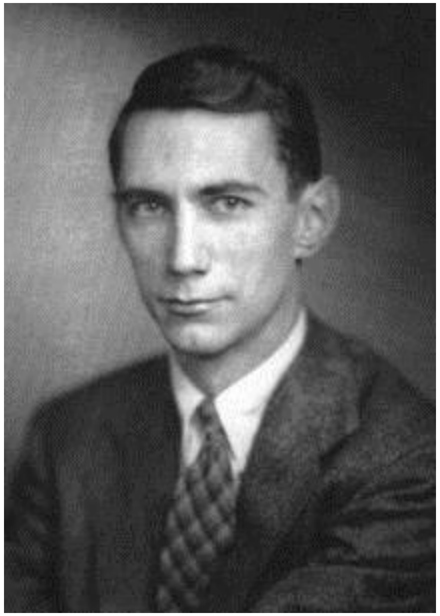
Lecture 4. Capacity of Fading Channels

- Capacity of AWGN Channels
- Capacity of Fading Channels
  - ✓ Ergodic Capacity
  - ✓ Outage Capacity

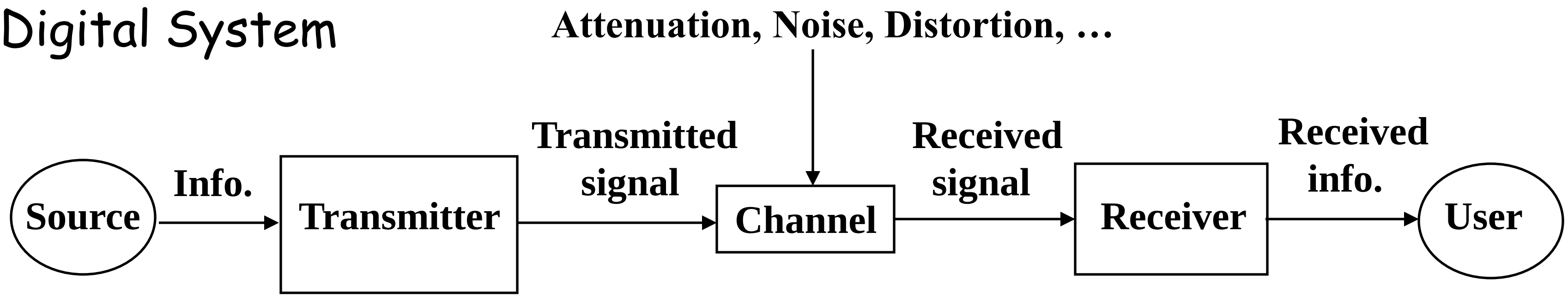
Shannon and Information Theory



Claude Elwood Shannon (April 30, 1916 – February 24, 2001)

Code Rate and Error Probability

Digital System



Information sequence

0000000
0000001
0000010
……
1111111

⇒

Codebook

00000000000
00000110011
00000101010
……
11111111111

Size: $M=2^k$

Code length: n>=k bits

What is the maximum rate R for which arbitrarily small P_e can be achieved?

- Error Prob.: $P_e = \Pr\{\hat{\mathbf{x}} \neq \mathbf{x}\}$
- Code Rate: $R=k/n=\log_2 M/n$

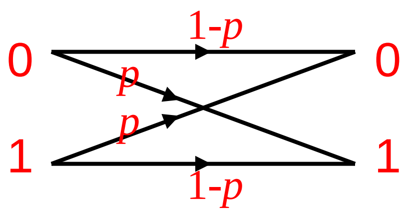
Channel Capacity: Binary Symmetric Channel

At the transmitter

000000000
000000101
000001010
000010111
……

Code length n

Discrete-memoryless Binary Symmetric Channel



At the receiver:

- For any codeword x_i, np bits will be received in error with high probability, if n is large.
- The number of possible error codewords corresponding to x_i is

$$\binom{n}{np} = \frac{n!}{(np)!(n(1-p))!} \approx 2^{nH_b(p)}$$

$$H_b(p) = -p\log_2 p - (1-p)\log_2(1-p)$$

Choose a subset of all possible codewords, so that the possible error codewords for each element of this subset is NOT overlapping!

- The maximum size of the subset: $M = \frac{2^n}{2^{nH_b(p)}} = 2^{n(1-H_b(p))}$
- The maximum rate that can be reliably communicated :

$$C = \frac{1}{n}\log_2 M = 1 - H_b(p)$$

(bit/transmission)

Channel Capacity: Discrete-time AWGN Channel

- $\mathbf{x}$ is an input sequence with power constraint: $\frac{1}{n}\sum_{i=1}^{n} x_i^2 \le P$

$$\mathbf{y} = \mathbf{x} + \mathbf{z}$$

- Noise z_i is a zero-mean Gaussian random variable with variance σ^2.

for large n

$$\frac{1}{n}\sum_{i=1}^{n} z_i^2 = \frac{1}{n}\sum_{i=1}^{n} (y_i - x_i)^2 \to \sigma^2$$

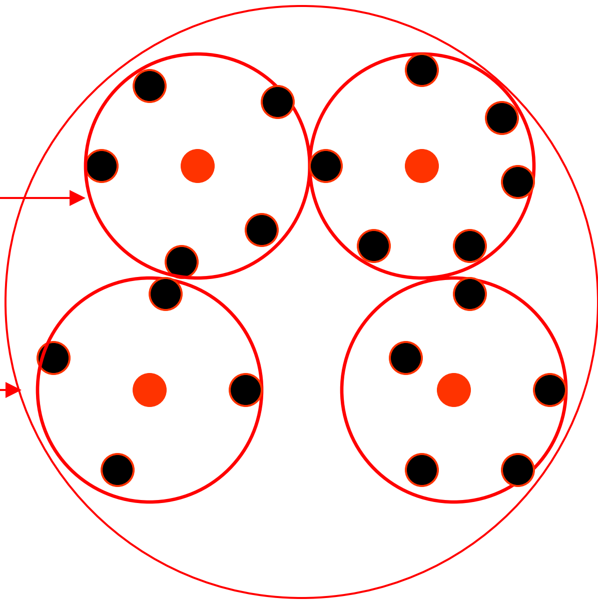
$$\|\mathbf{y}-\mathbf{x}\|^2 = n\sigma^2$$

n-dimensional hyper-sphere with radius $\sqrt{n\sigma^2}$

$$\frac{1}{n}\sum_{i=1}^{n} y_i^2 \le P + \sigma^2$$

$$\|\mathbf{y}\|^2 \le n(P+\sigma^2)$$

n-dimensional hyper-sphere with radius $\sqrt{n(P+\sigma^2)}$



- How many input sequences can we transmit over this channel at most such that the hyperspheres do not overlap?

$$M = (\sqrt{P+\sigma^2})^n / (\sqrt{\sigma^2})^n$$

- The maximum rate that can be reliably communicated :

$$C = \frac{1}{n}\log_2 M = \frac{1}{2}\log_2\left(1+\frac{P}{\sigma^2}\right)$$

(bit/transmission)

Channel Capacity: Continuous-time AWGN Channel

- Capacity of discrete-time AWGN channel:

$$C = \frac{1}{n}\log_2 M = \frac{1}{2}\log_2(1+\frac{P}{\sigma^2}) \quad \text{bit/transmission}$$

For continuous-time AWGN baseband channel with bandwidth W, power constraint $\boldsymbol{P}$ watts, and two-sided power spectral density of noise $N_0/2$,

- What is the average noise power per sampling symbol? $N_0 W$
- What is the minimum sampling rate without introducing distortion? $2W$

- Capacity of continuous-time AWGN channel:

$$C = 2W \cdot \frac{1}{2}\log_2(1+\frac{\boldsymbol{P}}{N_0 W}) = W\log_2(1+\frac{\boldsymbol{P}}{N_0 W}) \quad \text{bit/s}$$

More about Capacity of Continuous-time AWGN Channel

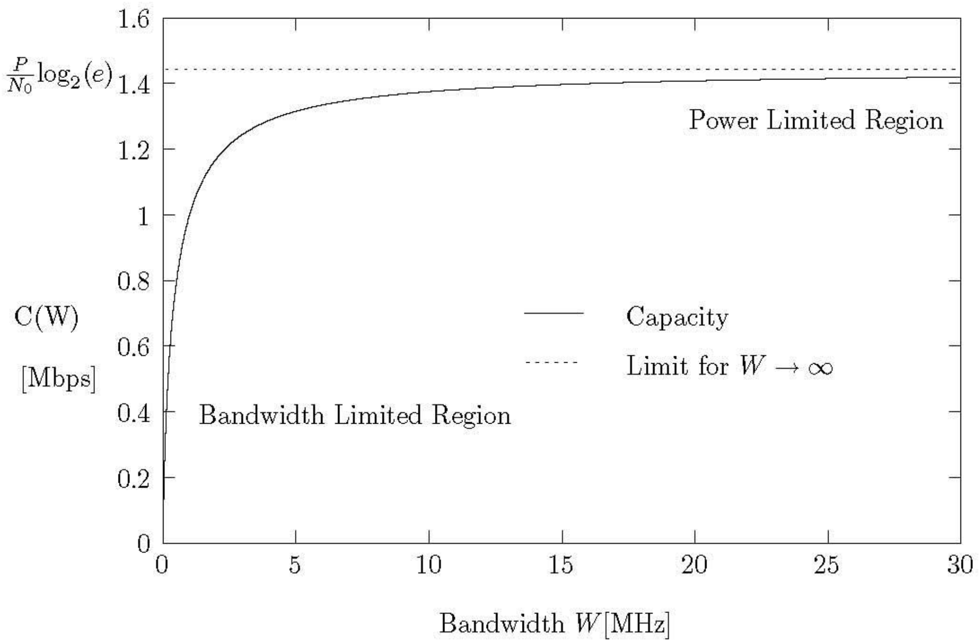
$$C = W \log_2(1 + \frac{P}{N_0 W}) \quad \text{bit/s}$$

- Can we increase the capacity by enhancing the transmission power?

  Yes, but the capacity increases logarithmically with P when P is large.

- If we can increase the bandwidth without limit, can we get an infinitely large channel capacity?

  No. $\lim_{W \to \infty} C = \frac{P}{N_0} \log_2 e$



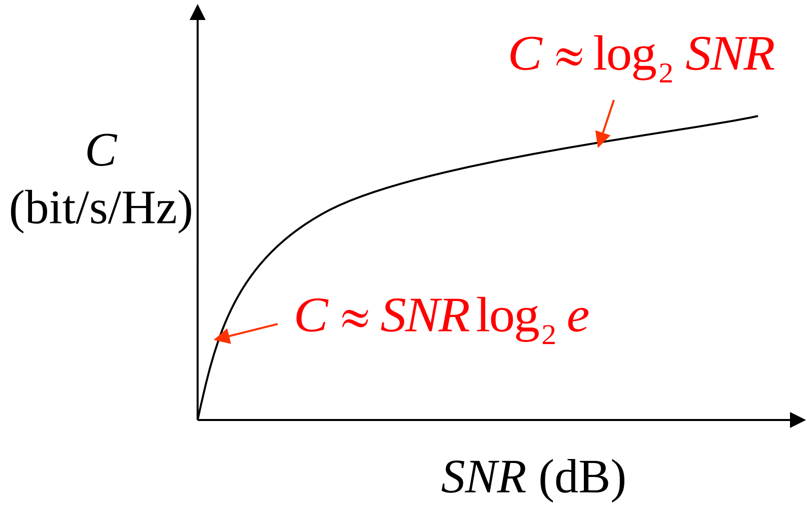
- How to achieve (approach) AWGN channel capacity?

  Turbo codes, LDPC, …?

More about Capacity of Continuous-time AWGN Channel

$$C = W\log_2\left(1+\frac{\mathbf{P}}{N_0 W}\right) \quad \text{bit/s}$$

- Spectral Efficiency: $C = \log_2(1+SNR)$ bit/s/Hz



$$SNR = \frac{\mathbf{P}}{N_0 W} = \frac{P}{N_0}$$

P: power per unit bandwidth

Capacity of Fading Channels I: Ergodic Capacity

- Capacity without CSIT
- Capacity with CSIT

Channel Model

- Consider a flat fading channel.
- Suppose each codeword spans L coherence time periods.
  - Without CSIT: the transmission power at each coherence time period is a constant P.

$$P_l = P,\ l=1,\ldots,L.$$

  - With CSIT: different transmission power can be allocated to different coherence time periods according to CSI. The average power is P.

$$P_l = f(h_l),\ l=1,\ldots,L. \quad \frac{1}{L}\sum_{l=1}^{L} P_l = P.$$

- Suppose the receiver has CSI.

Ergodic Capacity without CSIT

At each coherence time period, the reliable communication rate is

$$\log_2(1+|h_l|^2\,SNR) \qquad SNR=\frac{P}{N_0}$$

The average rate is $\frac{1}{L}\sum_{l=1}^{L}\log_2(1+|h_l|^2\,SNR)$

For ergodic channel: $\lim_{L\to\infty}\frac{1}{L}\sum_{l=1}^{L}\log_2(1+|h_l|^2\,SNR)=\mathrm{E}_h[\log_2(1+|h|^2\,SNR)]$

- Ergodic capacity without CSIT: $C_e^{wo}=\mathrm{E}_h[\log_2(1+|h|^2\,SNR)]$

- How to achieve it?
  - ✓ AWGN capacity-achieving codes
  - ✓ Coding across channel states

(Codeword should be long enough to average out the effects of both noise and fading.)

Ergodic Capacity without CSIT

$$C_e^{wo} = \mathrm{E}_h[\log_2(1+|h|^2\, SNR)] \le \log_2(1+\mathrm{E}_h[|h|^2]SNR) = \log_2(1+SNR) = C_{AWGN}$$

At low SNR,

$$C_e^{wo} \approx \mathrm{E}_h[|h|^2\, SNR]\log_2 e = SNR\log_2 e \approx C_{AWGN}$$

At high SNR,

$$C_e^{wo} \approx \mathrm{E}_h[\log_2(|h|^2\, SNR)] = \log_2 SNR + \mathrm{E}_h[\log_2|h|^2] \approx C_{AWGN} + \mathrm{E}_h[\log_2|h|^2]$$

$< C_{AWGN}$ $\mathrm{E}_h[\log_2|h|^2] < 0$

What if the transmitter has full CSI?

Ergodic Capacity with CSIT

- For a given realization of the channel gains $h_1, \ldots, h_L$ at L coherence time periods, the maximum total rate is

$$\max_{P_1,\ldots,P_L} \frac{1}{L}\sum_{l=1}^{L}\log_2\left(1+\frac{P_l\,|h_l|^2}{N_0}\right)$$

Subject to: $\frac{1}{L}\sum_{l=1}^{L} P_l = P.$

$$\Downarrow$$

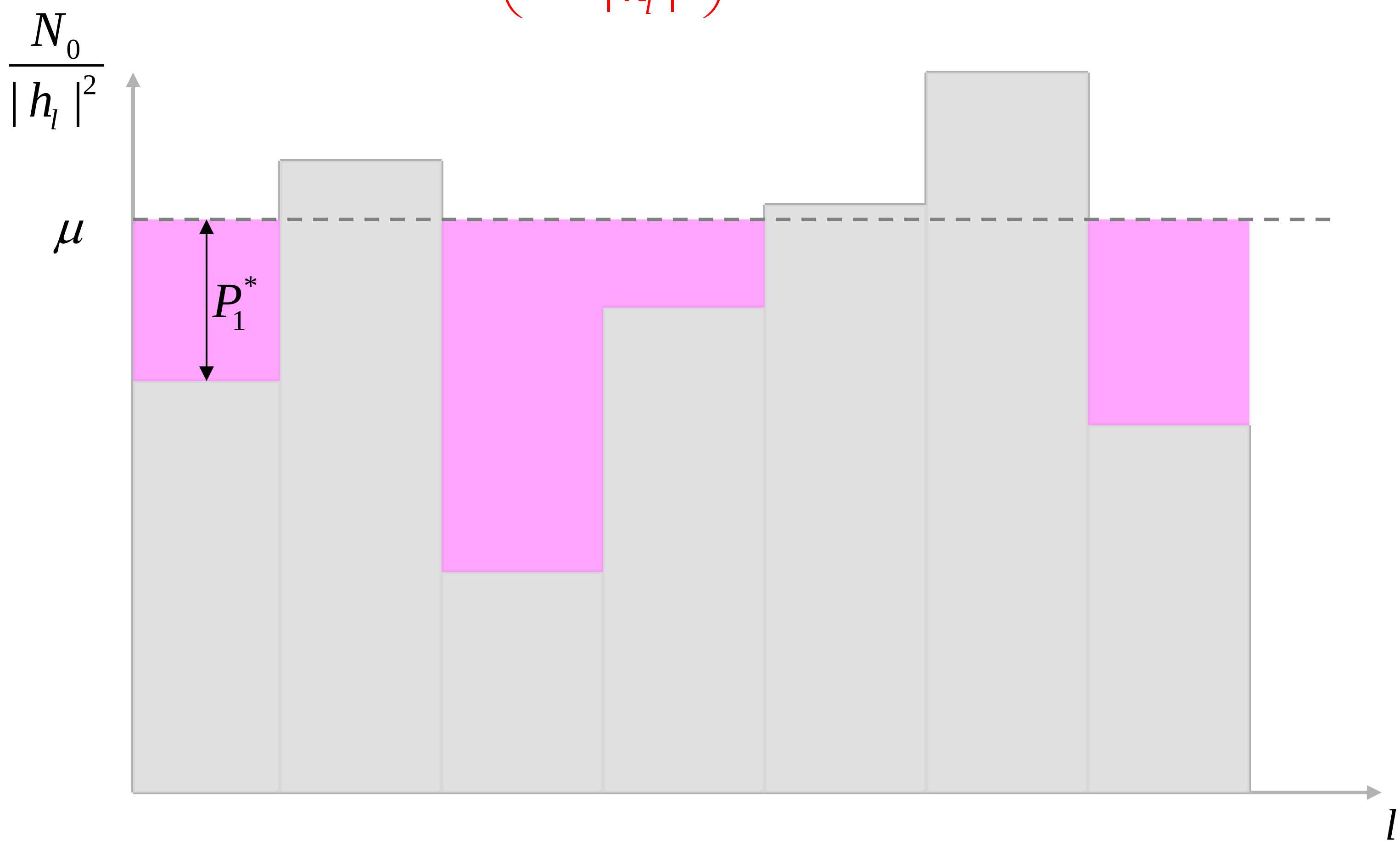
$$P_l^{optimal} = \left(\mu - \frac{N_0}{|h_l|^2}\right)^+.$$

Waterfilling Power Allocation

where μ satisfies: $\frac{1}{L}\sum_{l=1}^{L}\left(\mu - \frac{N_0}{|h_l|^2}\right)^+ = P$

Note: $x^+ = \begin{cases} x & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$

$$P_l^{optimal} = \left(\mu - \frac{N_0}{|h_l|^2} \right)^+, \quad l = 1, \ldots, L.$$



Ergodic Capacity with CSIT

As $L \to \infty,$

$$P = \mathrm{E}_h\left[\left(\mu - \frac{N_0}{|h|^2}\right)^+\right]$$

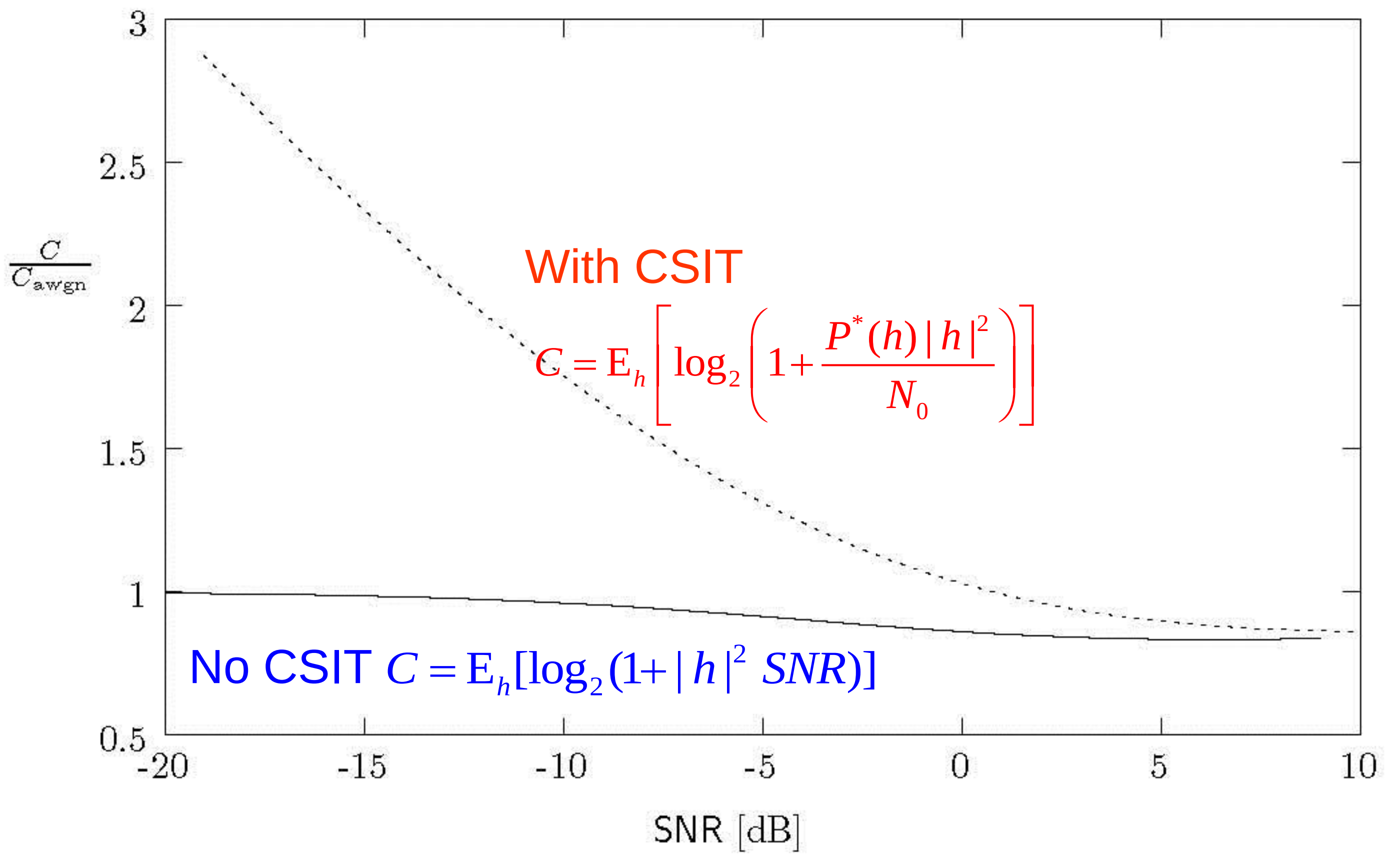
- μ converges to a constant which depends on the channel statistics (not the instantaneous channel realizations).
- The optimal power allocation is given by:

$$P^*(h) = \left(\mu - \frac{N_0}{|h|^2}\right)^+$$

- Ergodic capacity with CSIT:

$$C_e^w = \mathrm{E}_h\left[\log_2\left(1 + \frac{P^*(h)|h|^2}{N_0}\right)\right]$$

Ergodic Capacity with/without CSIT



Capacity of fading channel can be higher than that of AWGN channel if CSIT is available!

To Overcome Fading or to Exploit Fading?

- No CSIT:

$$C = \mathrm{E}_h[\log_2(1+|h|^2 \, SNR)]$$

  - Equal power allocation
  - Constant rate
  - Coding across channel states

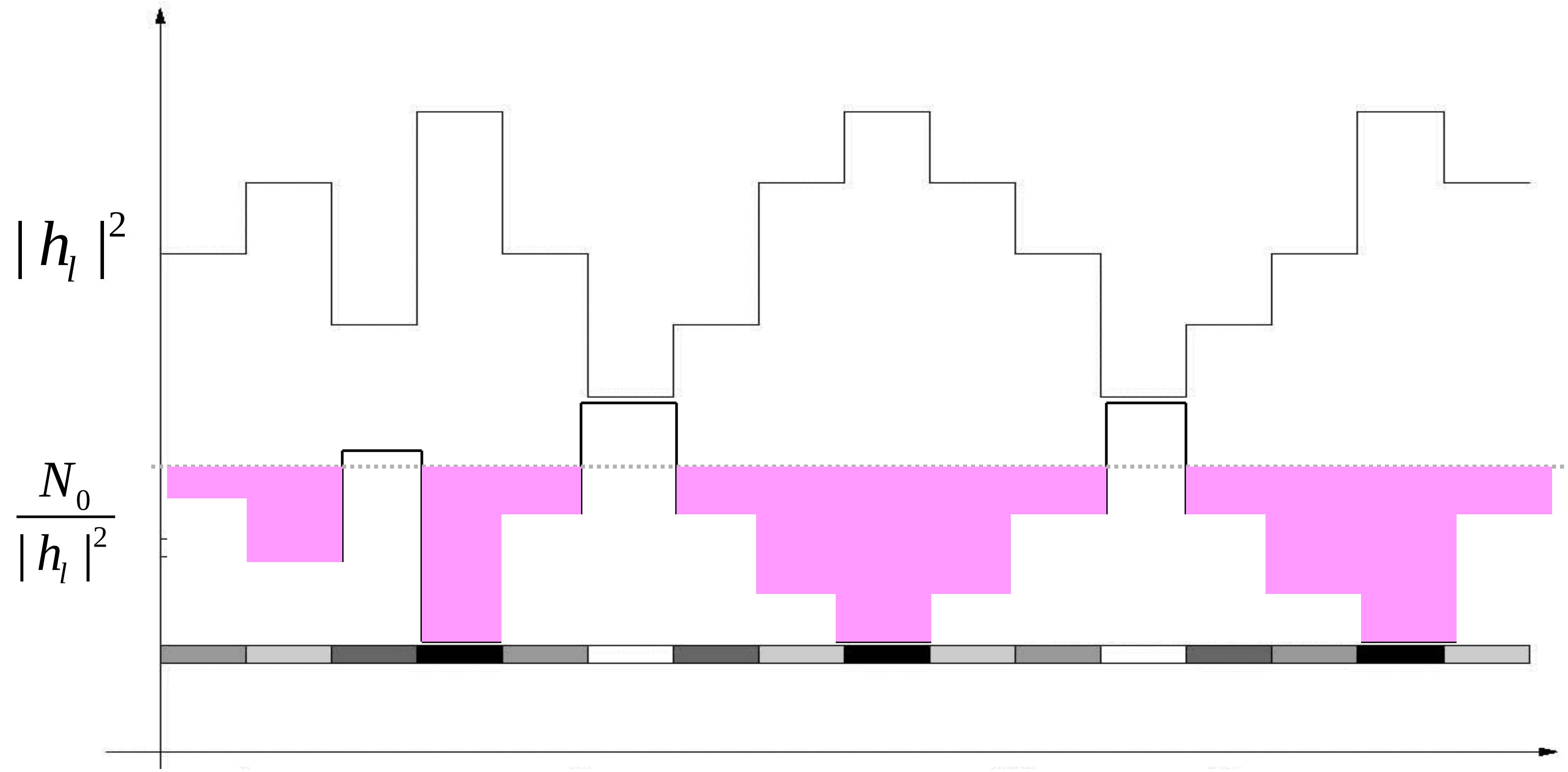
To average out fading!

- With CSIT:

$$C = \mathrm{E}_h\left[\log_2\left(1+\frac{P^*(h)|h|^2}{N_0}\right)\right]$$

How to exploit fading?

How to Exploit Fading?



- Waterfilling power allocation — Higher power at better subchannel
- Variable rate — Higher rate at better subchannel
- No coding across channel states

Summary I: Ergodic Capacity

- Without CSIT: to average out the fading effect
  - Close to AWGN channel capacity
  - Constant rate, coding across the channel states
- With CSIT: to exploit the fading effect
  - Higher than AWGN channel capacity at low SNR
  - Waterfilling power allocation + rate allocation

Capacity of Fading Channels II: Outage Capacity

- Capacity without CSIT
- Capacity with CSIT

Outage Event

- The ergodic capacity is based on the assumption that the channel is ergodic.

What if the ergodicity requirement is not satisfied?

- Suppose that:
  - CSI is available only at the receiver side.
  - The channel gain remains *unchanged* during the whole transmission.

Can we find a non-zero transmission rate R>0, below which reliable communication is guaranteed for any channel realization?

No. For any R>0, there always exists some channel realization h such that

$$\log_2(1+|h|^2\, SNR) < R \qquad \text{outage!}$$

Outage Probability

- Outage Probability $p_{out}(R)$: the probability that the system is in outage for given transmission rate R bit/s/Hz.

- Outage Probability without CSIT:

$$p_{out}^{wo}(R) = \Pr\{\log_2(1+|h|^2 SNR) < R\}$$

$$= \Pr\left\{|h|^2 < \frac{2^R - 1}{SNR}\right\} = F_{|h|^2}\left(\frac{2^R - 1}{SNR}\right) \qquad F_{|h|^2}(x) = \Pr\{|h|^2 \le x\}$$

  - For given rate R, the outage probability $p_{out}^{wo}(R)$ decreases as SNR increases.
  - For given SNR, the outage probability $p_{out}^{wo}(R)$ decreases as rate R decreases.
  - $p_{out}^{wo}(R)$ >0 for any SNR<∞ and R>0.

Outage Capacity

- Outage Capacity: $C_\varepsilon = \max\{R : p_{out}(R) \le \varepsilon\}$

- Outage Probability without CSIT: $p_{out}^{wo}(R) = \Pr\{\log_2(1+|h|^2 SNR) < R\}$
$$= F_{|h|^2}\left(\frac{2^R-1}{SNR}\right) = \varepsilon$$

- Outage Capacity without CSIT: $C_\varepsilon^{wo} = \log_2(1 + F_{|h|^2}^{-1}(\varepsilon) SNR)$

  – high SNR

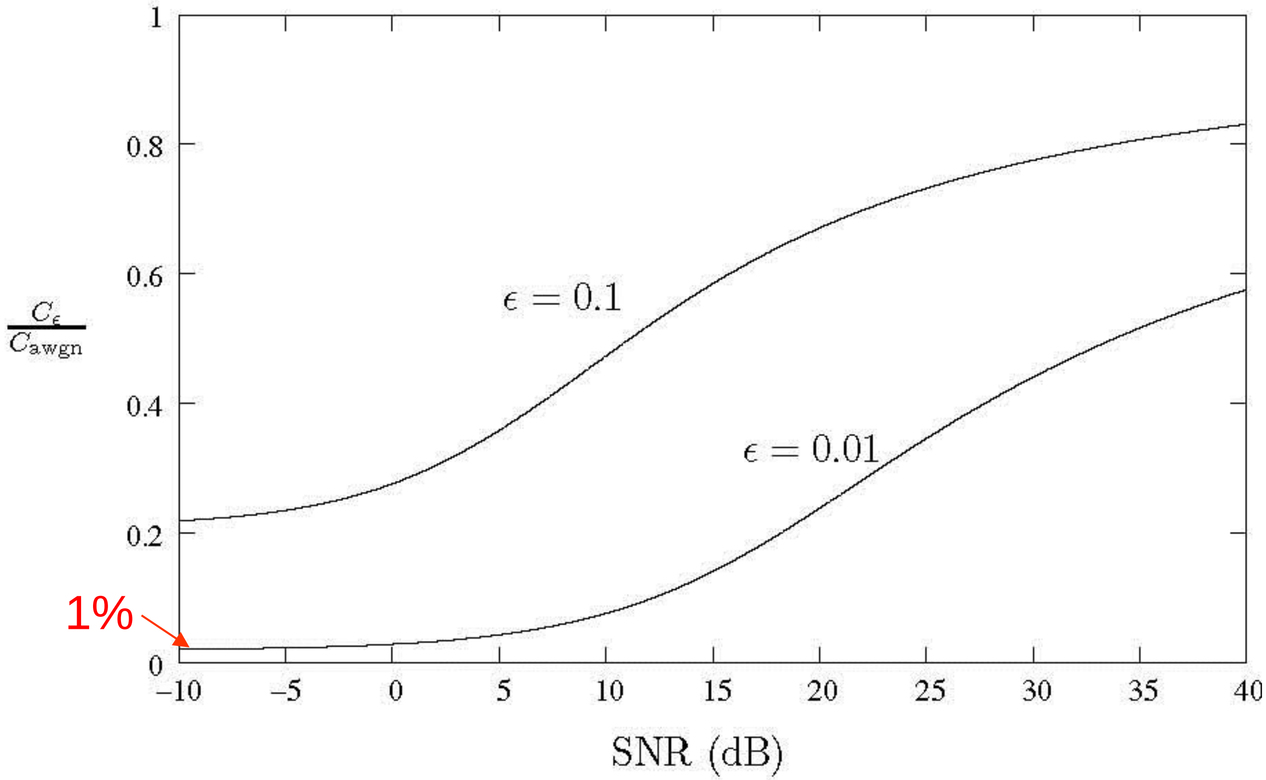
$$C_\varepsilon^{wo} \approx \log_2 SNR + \log_2 F_{|h|^2}^{-1}(\varepsilon)$$
$$\approx C_{AWGN} + \log_2 F_{|h|^2}^{-1}(\varepsilon)$$
<0

  – low SNR

$$C_\varepsilon^{wo} \approx F_{|h|^2}^{-1}(\varepsilon) SNR \log_2 e$$
$$\approx F_{|h|^2}^{-1}(\varepsilon) C_{AWGN}$$

How to improve the outage capacity?

Outage Capacity of Rayleigh Fading Channel



Outage Capacity with Receive Diversity

With L-fold receive diversity:

- Outage probability without CSIT:

$$p_{out}^{wo-rx}(R) = \Pr\{\log_2(1+\|\mathbf{h}\|^2\, SNR) < R\} = F_{\|\mathbf{h}\|^2}\left(\frac{2^R-1}{SNR}\right)$$

$$\approx \frac{(2^R-1)^L}{L!\,SNR^L}$$

Chi-square with 2L degrees of freedom: $F_{\|\mathbf{h}\|^2}(x) \approx \frac{x^L}{L!}$

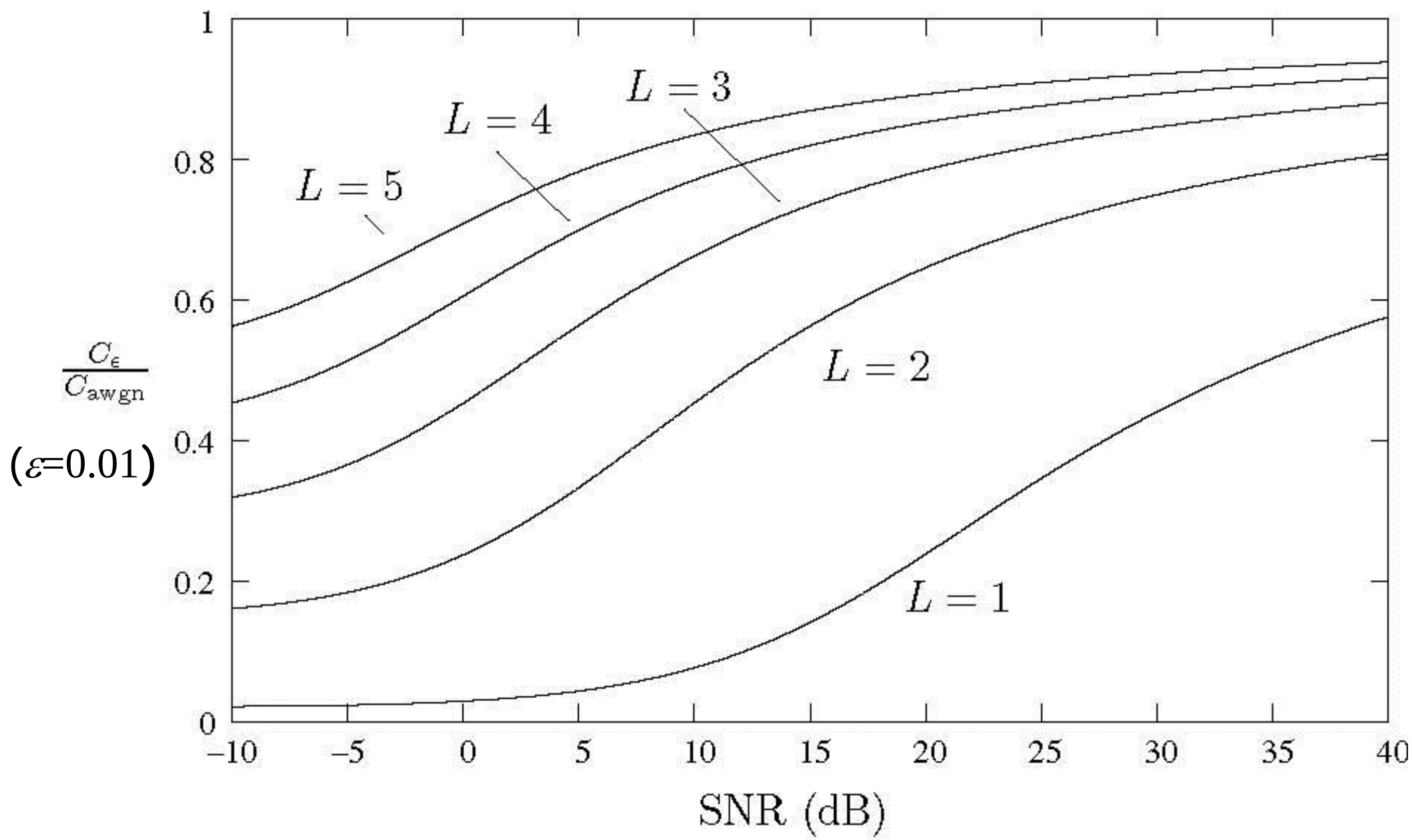
- Outage Capacity without CSIT:

$$C_\varepsilon^{wo-rx} = \log_2(1+F_{\|\mathbf{h}\|^2}^{-1}(\varepsilon)SNR) \approx \log_2(1+(\varepsilon L!)^{\frac{1}{L}}SNR)$$

– high SNR: $C_\varepsilon^{wo-rx} \approx \log_2 SNR + \frac{1}{L}\log_2(\varepsilon L!)$

– low SNR: $C_\varepsilon^{wo-rx} \approx (\varepsilon L!)^{1/L} SNR \log_2 e$

Outage Capacity with L-fold Receive Diversity



Outage Capacity with CSIT

- Suppose that:
  - CSI is available at **both** the receiver and the **transmitter** sides.
  - The channel gain remains *unchanged* during the whole transmission.

- With CSIT: $C_\varepsilon^w = \max\{R : \min_{P(h):\mathrm{E}_h[P(h)]=P} p_{out}(R, P(h)) \le \varepsilon\}$

The optimal power allocation strategy:

$$P^*(h) = \frac{P}{|h|^2\, \mathrm{E}_h[1/|h|^2]}$$

Channel inversion

Outage capacity with CSIT:

$$C_\varepsilon^w = \log_2\left(1 + \frac{SNR}{\mathrm{E}_h[1/|h|^2]}\right)$$

Delay-limited capacity
(zero-outage capacity)

Summary II: Outage Capacity

- Outage capacity is a useful metric for delay-sensitive scenarios.
  - Constant transmission rate
  - Usually much lower than AWGN channel capacity
    - Without CSIT: outage is unavoidable.
    - With CSIT: zero-outage is achieved by adjusting the transmission power inversely proportional to the channel gain.