Review of
Continuous-Time Signals and Systems

EE4015 Digital Signal Processing

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Prerequisites

- **Mathematic Pre-cursors:**
  - MA2001 (Multi-variable Calculus and Linear Algebra)
- **Signals and Systems Prerequisites:**
  - EE3210 (Signals and Systems)

Background Needed for DSP

- Review of Key Mathematic Concepts for Signal Processing
- Classification of Signals
  - Continuous-Time/Discrete-Time, Periodic/Non-Periodic, Deterministic/Non-Deterministic
- Basic deterministic signals for signal processing
  - Unit Impulse, Unit Step, Real Exponential, Complex Experiential and Sinusoidal Signals
- Continuous-Time Systems:
  - Causal, BIBO Stable, Linear, Time-Invariant, LTI System
  - Impulse Response
  - Convolution and its properties
  - Stability of LTI System

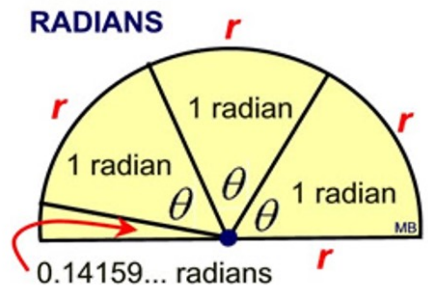
High School Algebra Review

Important Numbers for Signal Processing

- **1** : The number **one** is an identity for multiplication operations
  - $x * 1 = x$
- **0** : The number **zero** is an identity for addition operations
  - $x + 0 = x$
- ∞: This a symbol to represent an **infinite** number
  - $1/0 = \infty$
- $\boldsymbol{\pi}$: The **pi** number is the ratio of the circumference of any circle to the diameter of that circle.
  - $\pi = 3.14159265359$
  - *Circumference of a circle with radius 1 =* 2π

Angle Measure in Radians

- A radian is an angle made at the center of circle by an arc which is equal to the length of the radius of that circle.
  - It is therefore a unit that is used to measure an angle.
  - The one radian angle is approximately to 57.3°.



$$360^\circ = 2\pi\ rad$$

$$180^\circ = \pi\ rad$$

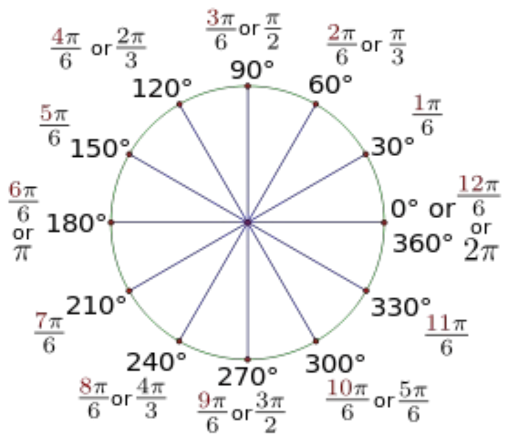
$$90^\circ = \frac{\pi}{2} rad$$

$$30^\circ = \frac{30}{180}\pi = \frac{\pi}{6} rad$$

$$45^\circ = \frac{45}{180}\pi = \frac{\pi}{4} rad$$

$$60^\circ = \frac{60}{180}\pi = \frac{\pi}{3} rad$$

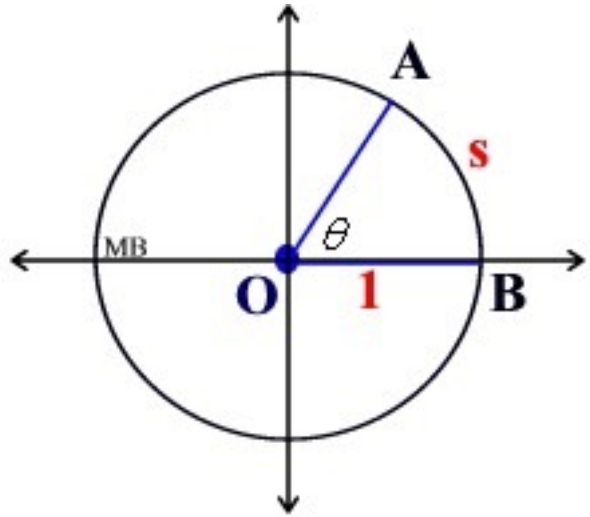
$$270^\circ = \frac{270}{180}\pi = \frac{27}{18}\pi = 1\frac{1}{2}\pi\ rad$$



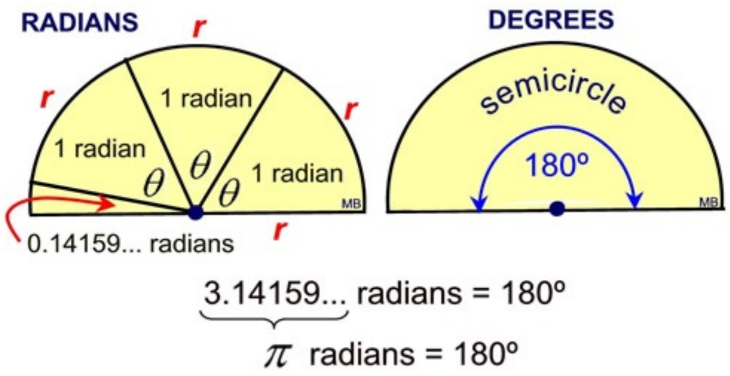
Angle Measure in Radians

- **A radian** is the measure of an angle θ that, when drawn as a central angle, subtends an arc whose length equals the length of the radius of the circle.
  - When working in the unit circle, with radius 1, the length of the arc equals the radian measure of the angle.

$$\underbrace{\theta = \frac{s}{r} = \frac{s}{1} = s}_{\theta = s}$$



- Relationship between Degrees and Radians



$360^o = 2\pi$
$180^o = \pi$
$90^o = \pi/2$

$60^o = \pi/3$
$45^o = \pi/4$
$30^o = \pi/6$

https://mathbitsnotebook.com/Algebra2/TrigConcepts/TCRadianMeasure.html

Geometric Series

$$a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{N-1} = \sum_{k=0}^{N-1} ar^k = a\left(\frac{1 - r^N}{1 - r}\right)$$

- $For\ |r| < 1\ and\ N = \infty,$

$$\sum_{k=0}^{\infty} ar^k = \frac{a}{1 - r}$$

- For example

$$1 + 0.5 + 0.5^2 + 0.5^3 + 0.5^4 + \cdots = \sum_{k=0}^{\infty} 0.5^k = \frac{1}{1-0.5} = 2$$

Exponential Series e^x

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\cdots$$

Leonhard Euler

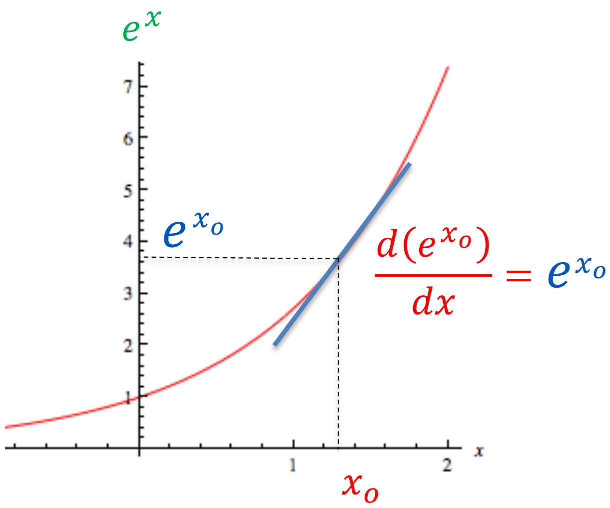
- $For = 1, e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots\cdots$

  e = 2.71828182845904523536 (and more ...)
- Euler's number ***e*** is a mathematical constant and it is named after the Swiss mathematician Leonhard Euler.
- There are many ways of calculating the value of e, another expression is

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n$$

Exponential Function of f(x)= e^x

$$f(x) = e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$$



- The exponential function is the solution of the simplest 1st order Ordinal Differential Equation (ODE) of

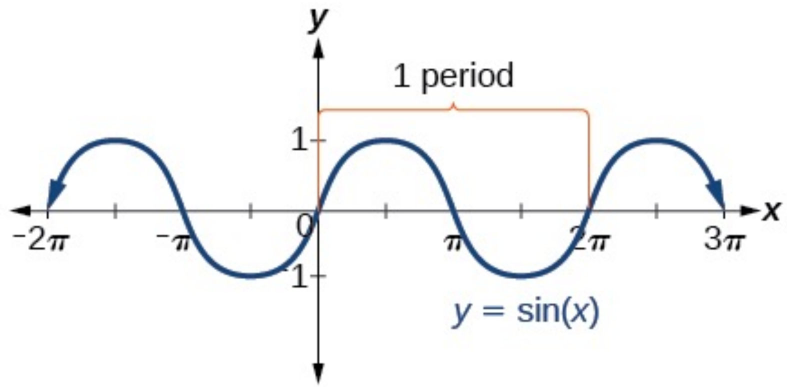
$$\frac{df(x)}{dx} = f(x) \Leftrightarrow \frac{d(e^x)}{dx} = e^x$$

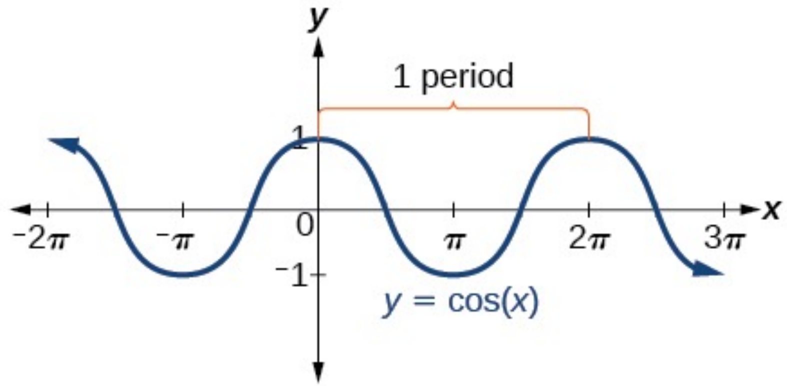
- This means the slop of the exponential function at x_o is equal to itself e^{x_o}
  - The growth rate of the value of the exponential function is proportional to the function's current value.

https://www.youtube.com/watch?v=oo1ZZlvT2LQ

Sine and Cosine Functions

- They are periodic function with period of 2π
  - $\sin(x + n2\pi) = \sin(x)$
  - $\cos(x + n2\pi) = \cos(x)$
  - $\cos\left(x + \frac{\pi}{2}\right) = \sin(x)$
- **Trigonometric Identities:**
  - $\sin^2(x) + \cos^2(x) = 1$
  - $\sin(x + y) = \sin(x)\cos(y) + \cos(y)\sin(x)$
  - $\cos(x + y) = \cos(x)\cos(y) - \sin(y)\sin(x)$





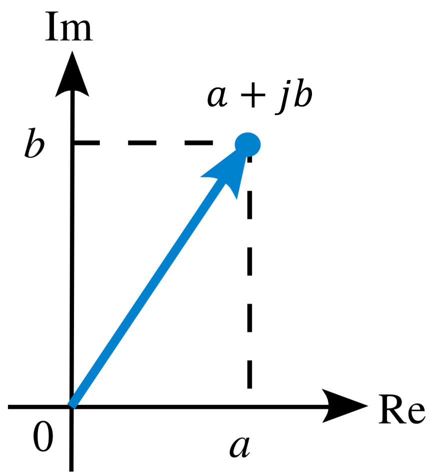
Complex Numbers : Rectangular Form

- A complex number x can be represented in rectangular form:

$$a + b\,j$$

Real Part **Imaginary Part**

$$j = \sqrt{-1}$$
$$j^2 = -1$$

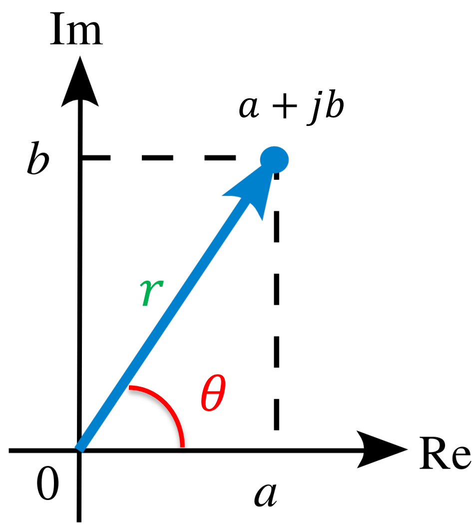


- Addition: $(a + jb) + (c + jd) = (a + c) + j(b + d)$
- Multiplication: $(a + jb)\,.\,(c + jd) = (ac - bd) + j(ad + bc)$
- Complex conjugate: $x^* = a - jb$

Complex Numbers : Polar Form

- A complex number x can also be represented in polar form as

$$x = a + jb = re^{j\theta}$$



- **Magnitude** : $r = |x| = \sqrt{a^2 + b^2}$
- **Phase** : $\theta = \tan^{-1}(b/a)$
- Polar Form Multiplication:

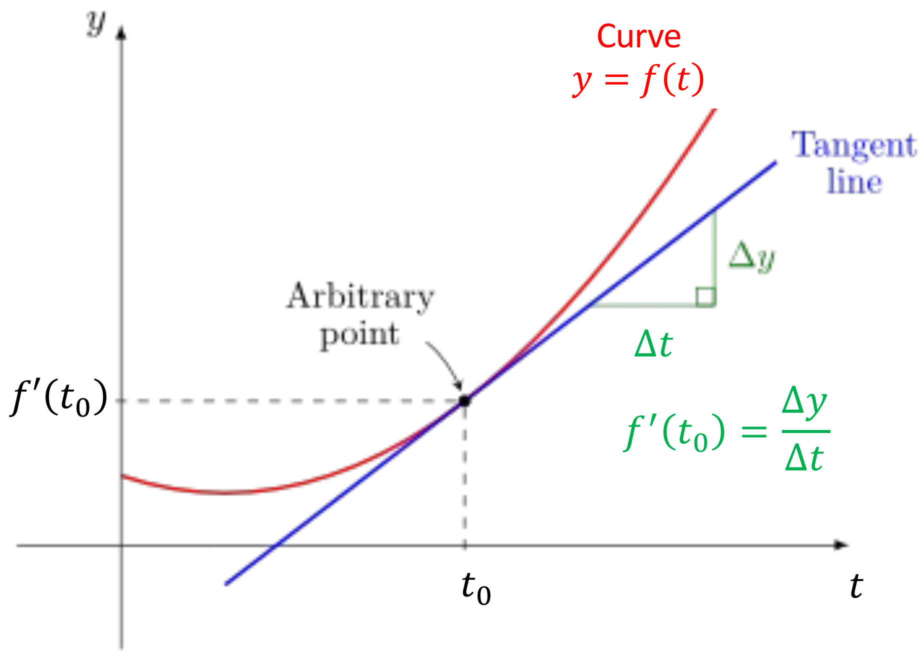
$$x \cdot y = |x|e^{j\theta_x} \cdot |y|e^{j\theta_y} = |x||y|e^{j(\theta_x + \theta_y)}$$

$$x \cdot x^* = |x|e^{j\theta_x} \cdot |x|e^{-j\theta_x} = |x|^2$$

Calculus Review

Highlights of Calculus

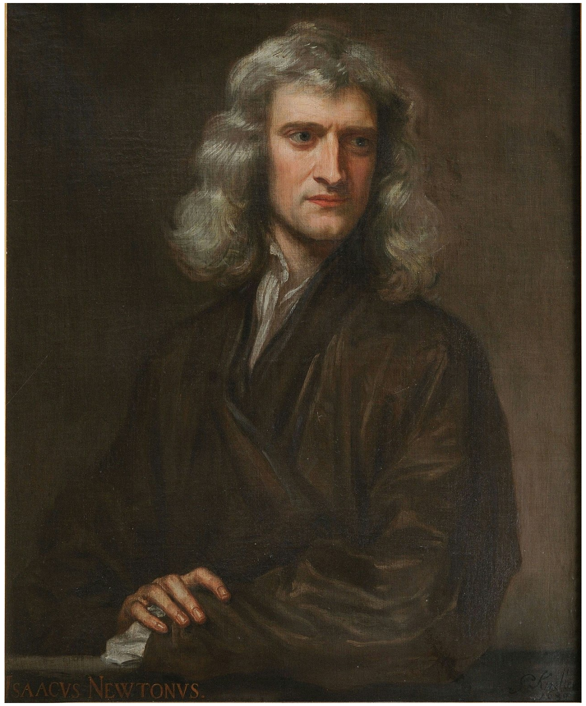
- What is **Differential Calculus** about?
  - **Rate of change**



- The derivative of a function $f(t)$,
  - $f'(t) = \frac{\mathrm{d}f(t)}{\mathrm{d}t} = \lim_{\Delta \to 0} \frac{f(t+\Delta)-f(t)}{\Delta}$
  - This function tells how quickly the function $f(t)$ is changing
  - $f'(t)$ is a function that describes the rate of change (or slop) the function $f(t)$

Newton's Law of Motion

- Newton's second law of motion:
  - Force is equal to rate of change of momentum (mass x velocity)
  - $\mathbf{F} = \frac{\mathrm{d}}{\mathrm{d}t}(m\boldsymbol{v}) = m\frac{\mathrm{d}^2\boldsymbol{x}}{\mathrm{d}t^2}$
  - $\mathbf{F} = m\boldsymbol{a}$



Isaac Newton (1643–1727), the physicist who formulated the laws of motion and invented Calculus

https://en.wikipedia.org/wiki/Newton%27s_laws_of_motion

Calculus – Derivative Table

Derivatives of Constant

$$f(x)=c \longrightarrow f'(x)=\frac{d}{dx}f(x)=0, where\, c=\text{constant}$$

Derivatives of Polynomials

$$f(x)=a\cdot x^n \longrightarrow f'(x)=\frac{d}{dx}f(x)=a\cdot n\cdot x^{n-1}$$

Derivatives of Exponential and Logarithmic functions

$$f(x)=c^{ax} \longrightarrow f'(x)=\frac{d}{dx}f(x)=c^{ax}\ln(c\cdot a), where\, c>0$$

$$f(x)=e^x \longrightarrow f'(x)=\frac{d}{dx}f(x)=e^x$$

$$f(x)=\log_c x \longrightarrow f'(x)=\frac{d}{dx}f(x)=\frac{1}{x\ln c}, where\, c>0, c\neq 1$$

$$f(x)=\ln x \longrightarrow f'(x)=\frac{d}{dx}f(x)=\frac{1}{x}, where\, x\neq 0$$

$$f(x)=\ln|x| \longrightarrow f'(x)=\frac{d}{dx}f(x)=\frac{|x|}{x^2}, where\, x\neq 0$$

$$f(x)=x^x \longrightarrow f'(x)=\frac{d}{dx}f(x)=x^x(1+\ln x)$$

Derivatives of Trigonometric Functions

$$f(x)=\sin(x) \longrightarrow f'(x)=\frac{d}{dx}f(x)=\cos(x)$$

$$f(x)=\cos(x) \longrightarrow f'(x)=\frac{d}{dx}f(x)=-\sin(x)$$

$$f(x)=\tan(x) \longrightarrow f'(x)=\frac{d}{dx}f(x)=\sec^2(x)=\frac{1}{\cos^2(x)}=1+\tan^2(x)$$

$$f(x)=\sec(x) \longrightarrow f'(x)=\frac{d}{dx}f(x)=\sec(x)\tan(x)$$

$$f(x)=\csc(x) \longrightarrow f'(x)=\frac{d}{dx}f(x)=-\csc(x)\cot(x)$$

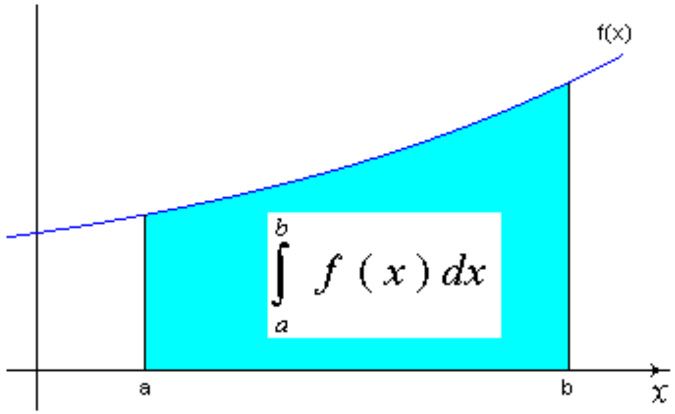
$$f(x)=\cot(x) \longrightarrow f'(x)=\frac{d}{dx}f(x)=-\csc^2(x)=\frac{-1}{\sin^2(x)}=-(1+\cot^2(x))$$

Integral Calculus

- Just the **inverse** of differential calculus, given a function $f(t)$ to find functions $g(t)$ with rate of change described by $f(t)$
  - $g(t) = \int f(t)dt$ => $\frac{\mathrm{d}f(t)}{\mathrm{d}t} = g(t)$
  - $g(t) = \int t\, dt = \frac{1}{2}t^2 + C$

Definite Integrals

- **Definite integrals** can be interpreted formally as the signed area of the region in the plane that is bounded by the graph of a given function between two points in the real line.
- $\int_a^b f(t)\,dt$
- $\int_1^4 t\,dt = \left[\frac{1}{2}t^2\right]_1^4 = \frac{1}{2}4^2 - \frac{1}{2}1^2 = 8 - \frac{1}{2} = 7.5$



Calculus – Integrals Table

1. $\int x^n \, dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$

2. $\int \frac{1}{x} \, dx = \ln|x|$

3. $\int e^x \, dx = e^x$

4. $\int b^x \, dx = \frac{b^x}{\ln b}$

5. $\int \sin x \, dx = -\cos x$

6. $\int \cos x \, dx = \sin x$

7. $\int \sec^2 x \, dx = \tan x$

8. $\int \csc^2 x \, dx = -\cot x$

9. $\int \sec x \tan x \, dx = \sec x$

10. $\int \csc x \cot x \, dx = -\csc x$

11. $\int \sec x \, dx = \ln|\sec x + \tan x|$

12. $\int \csc x \, dx = \ln|\csc x - \cot x|$

13. $\int \tan x \, dx = \ln|\sec x|$

14. $\int \cot x \, dx = \ln|\sin x|$

17. $\int \frac{dx}{x^2 + a^2} = \frac{1}{a} \tan^{-1}\left(\frac{x}{a}\right)$

18. $\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\left(\frac{x}{a}\right), \quad a > 0$

Euler's Formula

- Based on Taylor Series

$$f(x) = f(a) + \frac{f'(a)}{1!}(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!} + \cdots + \frac{f^n(a)}{n!}(x-a)^n$$

- We have following power series of $f(x) = e^{jx}$ with use of a=0 and all derivative $f^n(0) = j^n$

$$e^{jx} = 1 + jx - \frac{x^2}{2!} - j\frac{x^3}{3!} + \frac{x^4}{4!} + j\frac{x^5}{5!} - \frac{x^6}{6!} - j\frac{x^7}{7!} + \cdots$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad \sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$e^{jx} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right) + j\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$

$$\boxed{e^{jx} = \cos(x) + j\sin(x)}$$

Properties of Euler's Formula

$$e^{jx} = \cos(x) + j\sin(x) \qquad e^{-jx} = \cos(x) - j\sin(x)$$

$$\cos(x) = \frac{1}{2}\left(e^{jx} + e^{-jx}\right) \qquad \sin(x) = \frac{1}{2j}\left(e^{jx} - e^{-jx}\right)$$

$$\left|e^{\pm jx}\right| = \sqrt{\cos^2(x) + \sin^2(x)} = 1$$

$$\angle\, e^{\pm jx} = \tan^{-1}\left(\pm\frac{\sin(x)}{\cos(x)}\right) = \tan^{-1}(\pm\tan(x)) = \pm x$$

Euler's Identity

- Euler's identity (aka Euler's equation) is the equality

$$\boxed{e^{j\pi} + 1 = 0}$$

where *1* – One (Unity) is identity of multiplication operation,

0 – Zero (Nothing) is identity of addition operation,

j – Imaginary unit ($j = \sqrt{-1}$), which satisfies $j^2 = -1$,

π – pi (3.1415…), the ratio of the circumference of a circle to its diameter,

e – Euler's number (2.7182…), the base of natural logarithms.

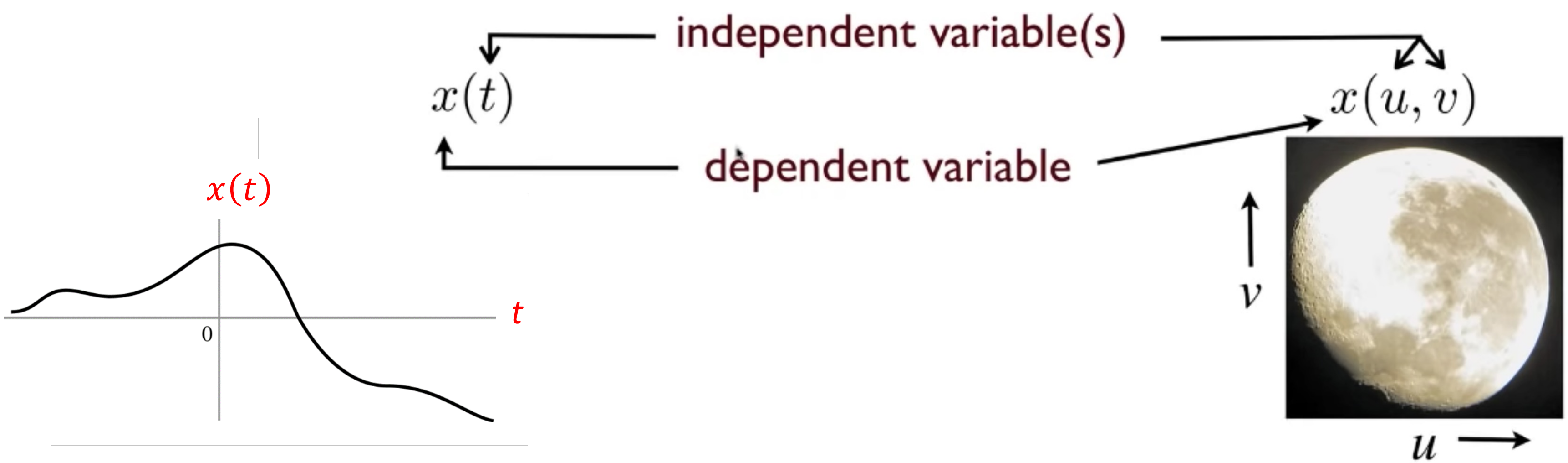
Euler's identity is named after the Swiss mathematician Leonhard Euler. It is considered to be an example of mathematical beauty.

Leonhard Euler

Continuous-Time Signals and Systems

Signals

- A **signal** describes how some physical quantity varies over time and /or space.
- Mathematically, a function of one or more variables.
  - For 1-D signals, the independent variable often represents ***time : t***
  - For 2-D signals, the independent variable often represents ***spatial position : (u, v)***

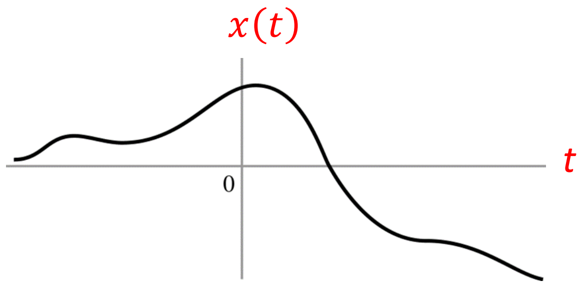


Classification of Signals

- Continuous-Time and Discrete-Time Signals
- Periodic and Non-Periodic (Aperiodic) Signals
- Deterministic and Non-Deterministic (Random) Signals
- Elementary continuous-time deterministic signals for Signal Processing
  - Impulse Signal (Delta Function) : $\delta(t)$
  - Unit Step Signal : $u(t)$
  - Exponential Signals : $C\ e^{at}$
  - Sinusoidal Signals : $A \cos(\Omega_o t + \theta)$

Continuous-Time Signals (Analog Signals)

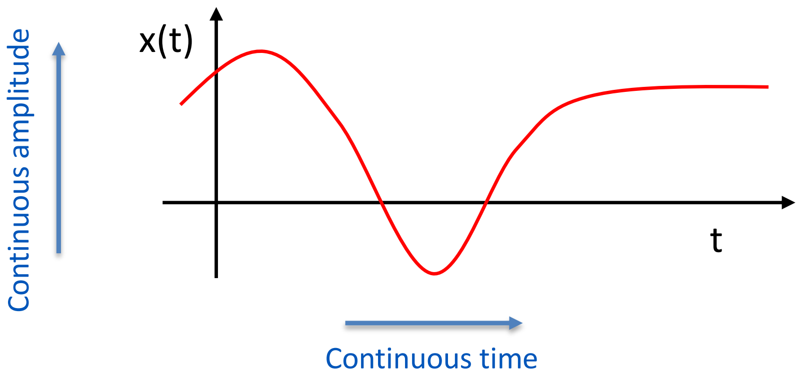
- Continuous-Time (CT) signal is a signal that exists at every instant of time
  - A CT signal is often referred to as analog signal
  - The independent variable is a continuous variable : t
  - Amplitude of CT signal can assume any value over a continuous range of numbers

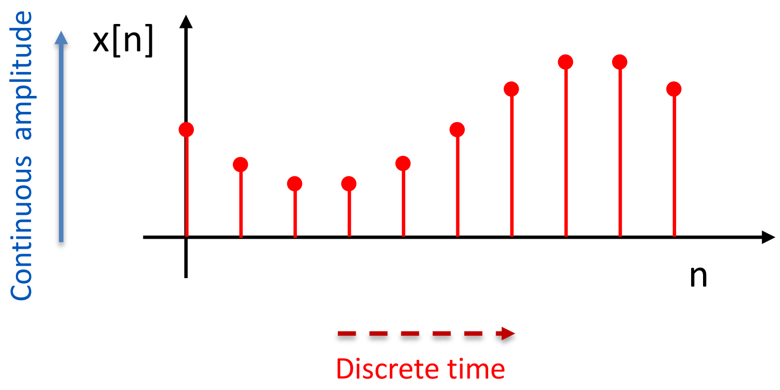


- Most of the signals in the physical world are CT signals.
- Examples: voltage & current, pressure, temperature, velocity, etc.

Discrete-Time Signals

- A signal defined only for discrete values of time is called a discrete-time (DT) signal x[n] or simply a sequence
- DT signal can be obtained by taking samples of an analog signal at discrete instants of time
- The values of each sample x[n] is **continuous**



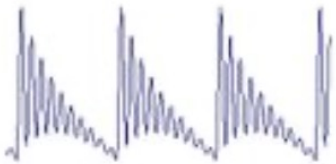


Periodic and Non-Periodic Signals

<u>Continuous-Time Signals</u>

Periodic Signals

(Single Sinewaves)



(Multiple Sinewaves)

$$\tilde{x}(t) = \tilde{x}(t + nT_o) \;\; for\ all\ t$$

Non-Periodic (Aperiodic) Signals

(Noise)

(Pulse)

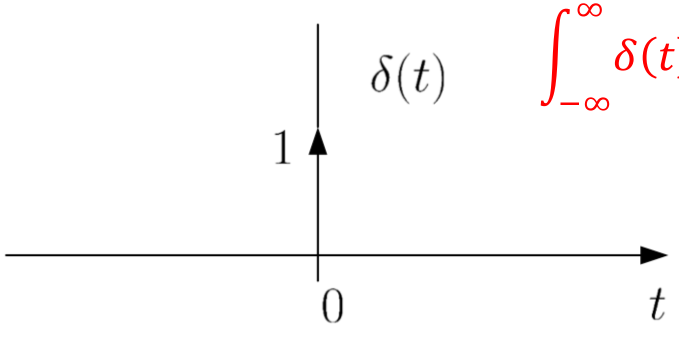
$$x(t)$$

Deterministic and Non-Deterministic Signals

- A signal is said to be deterministic if there is no uncertainty with respect to its value at any instant of time.
  - They can be described by a mathematical equation
- A non-deterministic signal is one where there is uncertainty at any instant of time.
  - Non-deterministic signals are random in nature hence they are called random signals.
  - Random signals cannot be described by a mathematical equation.
  - They are modelled by probabilistic and statistic tools.

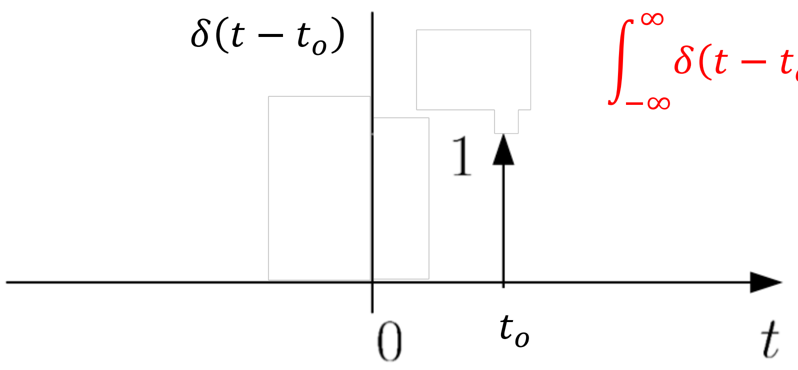
Impulse Signal (Dirac Delta Function) $\delta(t)$

- The impulse signal $\delta(t)$ has the value zero everywhere except at $x = 0$ where its value is infinitely large in such a way that its total integral is 1.
- The Dirac delta function $\delta(t)$ is very useful in many areas of physics. It is not an ordinary function, in fact properly speaking it can only live *inside* an integral.



$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1$$

$\delta(t)$ is a spike centered at $t = 0$



$$\int_{-\infty}^{\infty} \delta(t - t_o)\, dt = 1$$

$\delta(t - t_o)$ is a spike centered at $t = t_o$

Representation of CT Signals by Impulse Signal

- The product of the time-shifted impulse signal $\delta(t-t_o)$ with any CT signals is zero except where $t = t_o$
- Formally, for any CT signal $x(t)$, its amplitude at $t = t_o$ can be represented as integral of product between the signal $x(t)$ and time-shifted impulse signal $\delta(t-t_o)$:

$$\int_{-\infty}^{\infty} x(t)\,\delta(t-t_o)\,dt = x(t-t_o)$$

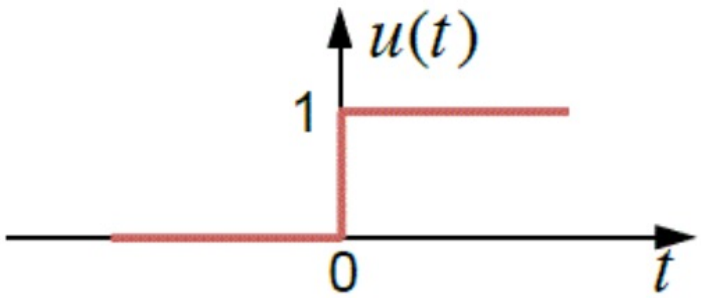
- **This is a very important equation for deviating the convolution concept for Linear Time-Invariant System.**

Unit Step Signal $u(t)$

- The unit step signal $u(t)$ has the form of :

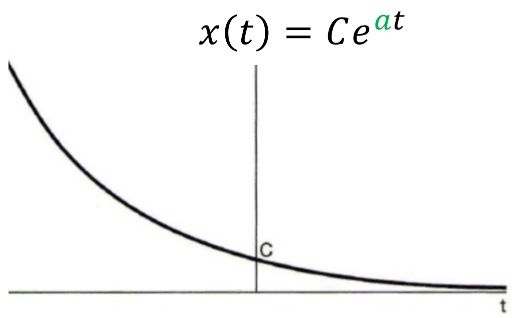
$$u(t) = \begin{cases} 1, & t > 0 \\ 0, & t < 0 \end{cases}$$

- As there is a sudden change from 0 to 1 at $t = 0$, $u(0)$ is not well defined

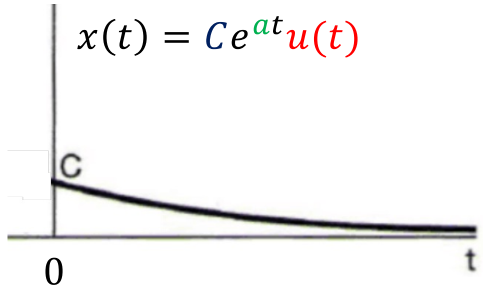


Causal and Non-Causal Signals

- A signal is causal if it is zero for $t < 0$
- Causal signals are readily created by multiplying any continuous signal by the unit step signal $u(t)$
  - $x(t) = Ce^{at}$: Non-Causal Signal



  - $x(t) = Ce^{at}u(t)$: Causal Signal
    - $x(t) = 0 \quad for\ t < 0$



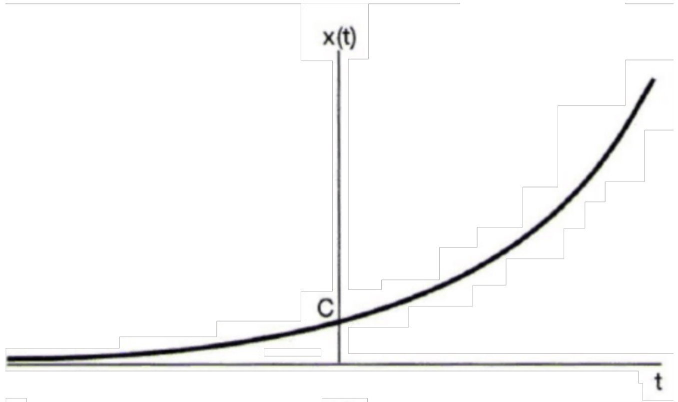
Real Exponential Signals

- General form

$$x(t) = C\ e^{at}$$

where C and a are complex numbers

- **Real Exponential Signals**
  - C and a are real
  - Support that $C > 0$
  - $a > 0$: $x(t)$ is growing exponential
  - $a < 0$: $x(t)$ is decaying exponential
  - $a = 0$: $x(t)$ is constant



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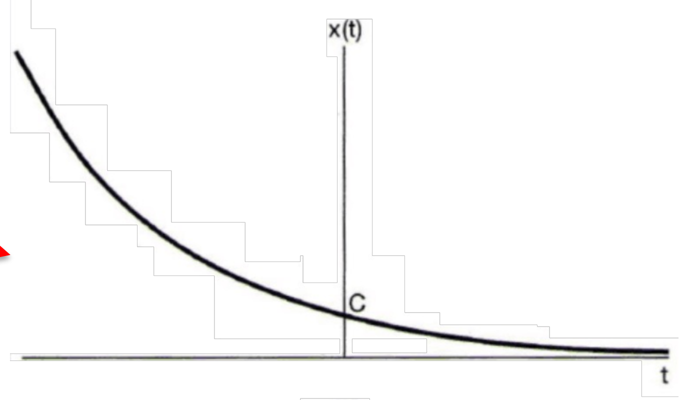


(b)

Complex Exponential Signals

- Periodic complex exponential signals
  - a is purely imaginary

$$x(t) = e^{j\Omega_o t}$$

- Characterization of the period T_o

$$e^{j\Omega_o t} = e^{j\Omega_o (t+T_o)} = e^{j\Omega_o t} e^{j\Omega_o T_o} = e^{j\Omega_o t}$$

Because $e^{j\Omega_o T_o} = 1$

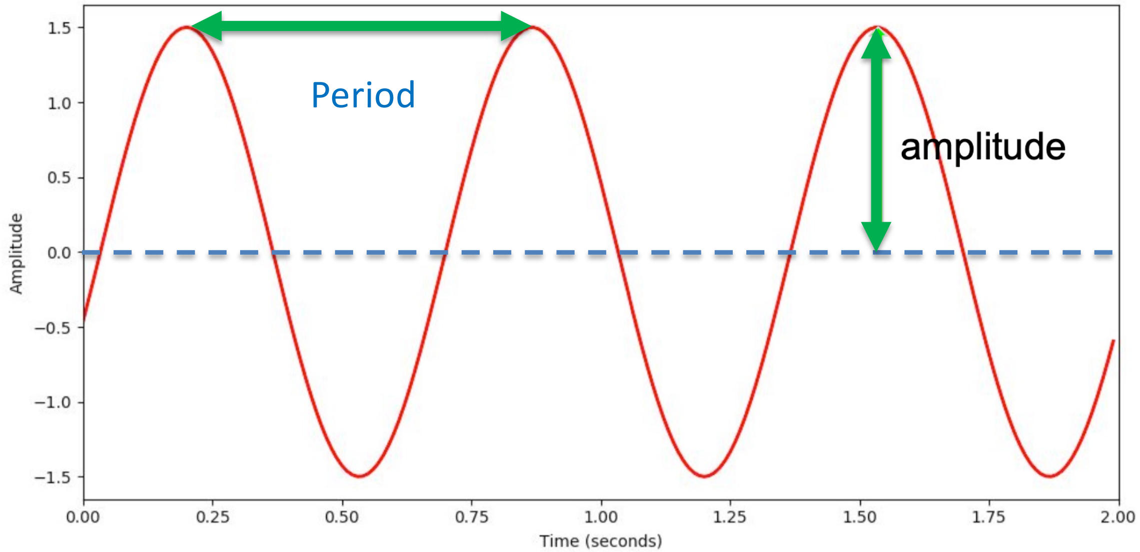
$$T_o = \frac{2\pi}{|\Omega_o|}$$

T_o is fundamental period

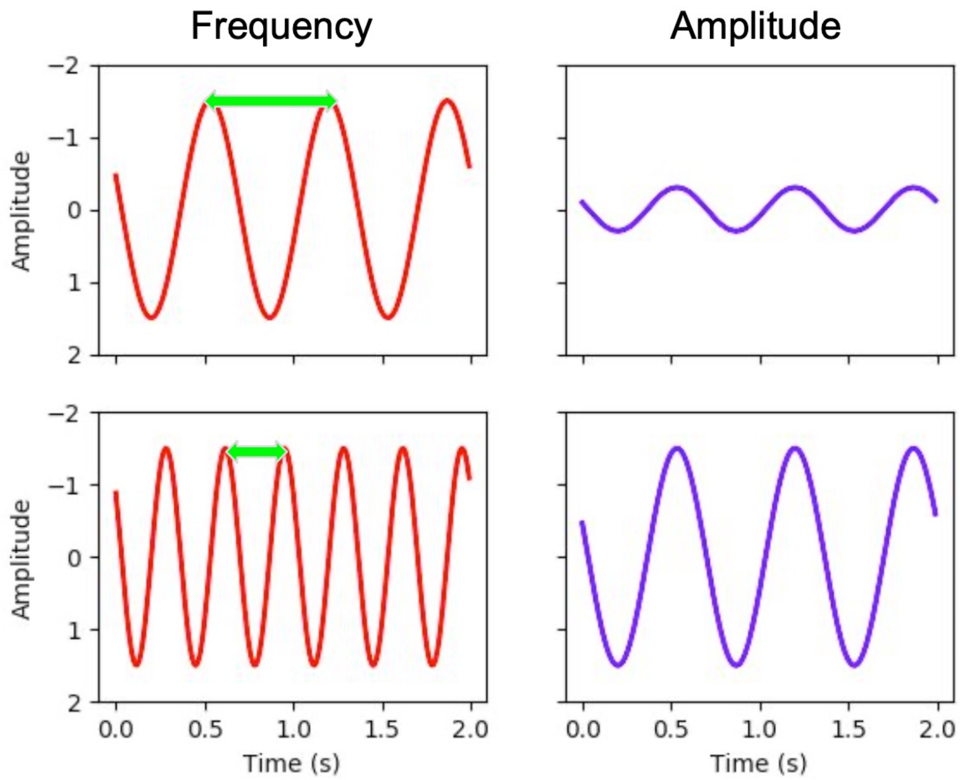
$|\Omega_o|$ is fundamental angular frequency

Continuous-Time Sinusoidal Signals

- General form of sine and cosine functions:
  - $y(t) = A\cos(2\pi f_o t + \theta) = A\cos(\Omega_o t + \theta)$
  - $f_o = \frac{1}{T_o}$ is the Frequency in Hz and T_o is the period
  - $\Omega_o = 2\pi f_o$ is the Angular Frequency in radians/sec
  - A is the Amplitude
  - θ is the phase angle in radians

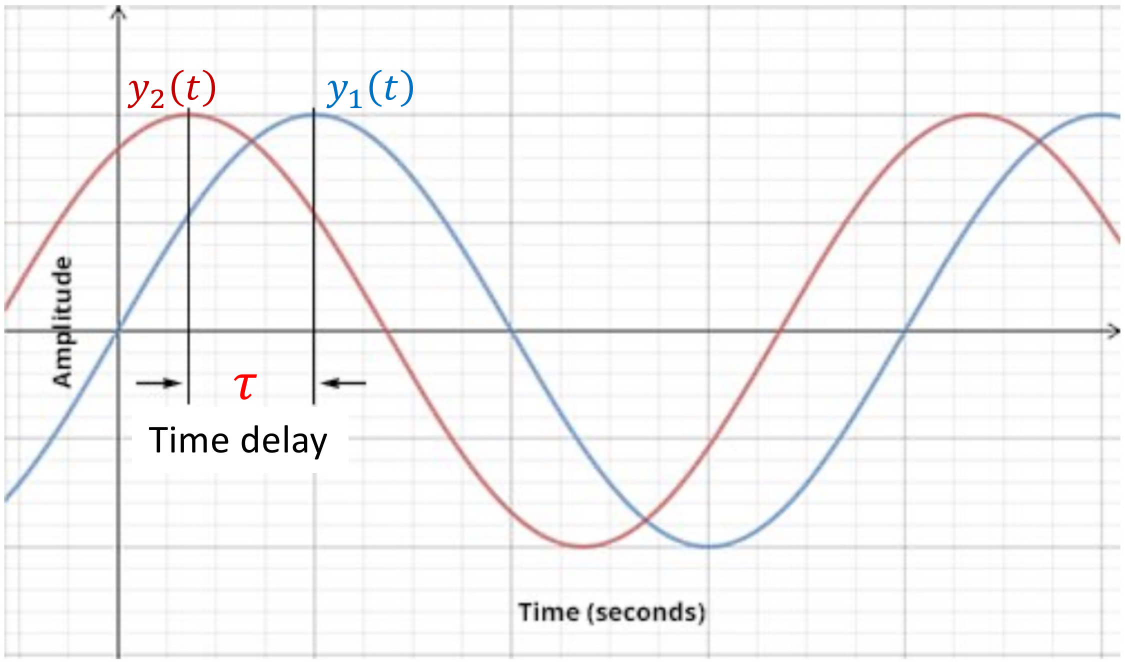


Frequency and Amplitude



Phase

- $y_1(t) = A\sin(\Omega_o t)$
- $y_2(t) = A\sin(\Omega_o t + \theta)$
- T_o is period
- Phase θ is in radians
- $\theta = 2\pi\frac{\tau}{T_o} = \tau\,\Omega_o$



Relation Between Complex Exponential and Sinusoidal Signals

- Based on Euler's Formula, complex exponential signal can be written in term of sinusoidal signals

$$x(t) = e^{j\Omega_o t} = \cos \Omega_o t + j \sin \Omega_o t$$

- Sinusoidal signal can be written in terms of two periodic complex exponentials

$$A\cos(\Omega_o t + \theta) = \frac{A}{2} e^{j\theta} e^{j\Omega_o t} + \frac{A}{2} e^{-j\theta} e^{-j\Omega_o t}$$

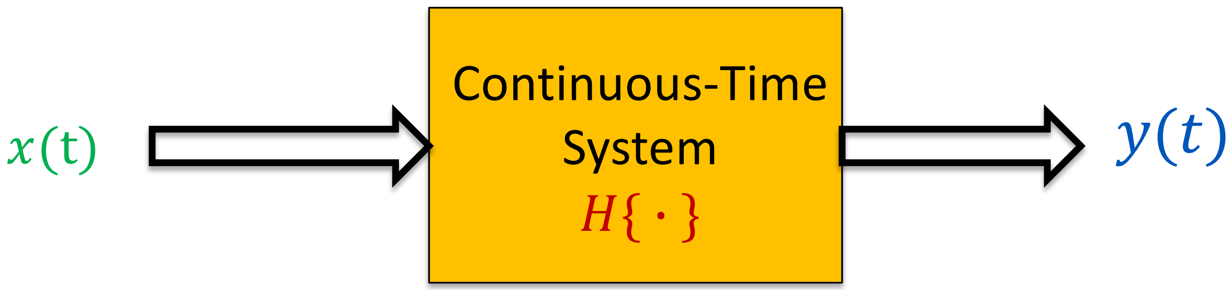
- Sinusoidal signal can be written in terms of a single periodic complex exponential

$$A\cos(\Omega_o t + \theta) = A\,Re\{e^{j\theta} e^{j\Omega_o t}\}$$

$$A\sin(\Omega_o t + \theta) = A\,Im\{e^{j\theta} e^{j\Omega_o t}\}$$

Continuous-Time Systems

- A continuous-time system is a transformation that operates on a continuous-time signal $x(t)$ called the input to produce another continuous-time signal $y(t)$.



$$y(t) = H\{x(t)\}$$

- where the symbol H denotes the transformation or processing performed by the system

Causal Systems

- A system is said to be causal if the output of the system at any time '*t*' depends only on present and past inputs but does not depend on future inputs.
- If a system does not satisfy this definition, it is called noncausal.
  - The noncausal systems have outputs that depend not only on present and past inputs but also on future inputs.

BIBO Stable System

- A continuous-time signal $x[n]$ is bounded if there exists a finite M such that
  - $|x(t)| < M$
- A Continuous-time system in Bounded Input-Bounded Output (**BIBO**) stable if every bounded input signal $x(t)$ produced a bounded output signal $y(t)$.
  - If the input $\underbrace{|x(t)| < A}_{\text{Bounded Input}}$, then the output $\underbrace{|y(t)| < B}_{\text{Bounded Output}}$

Linear Systems

- A Linear system is defined as follows:

$$x_1(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y_1(t)=H\{x_1(t)\}$$

$$x_2(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y_2(t)=H\{x_2(t)\}$$

- For arbitrary constants a_1 and a_2 :

$$a_1x_1(t) + a_2x_2(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y(t) = a_1y_1(t) + a_2y_2(t)$$

Time-Invariant Systems

- A time-invariant system is defined as follows:

$$x_1(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y_1(t) = H\{x_1(t)\}$$

$$x_1(t - t_o) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y(t) = H\{x_1(t - t_o)\} = y_1(t - t_o)$$

- Specifically, a system is time invariant if a time shift in the input signal results in an identical time shift in the output signal.

Linear Time-Invariant (LTI) Systems

- LTI systems satisfy both Linear and Time-Invariant properties.

$$x_1(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y_1(t)=H\{x_1(t)\}$$

$$x_2(t) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow y_2(t)=H\{x_2(t)\}$$

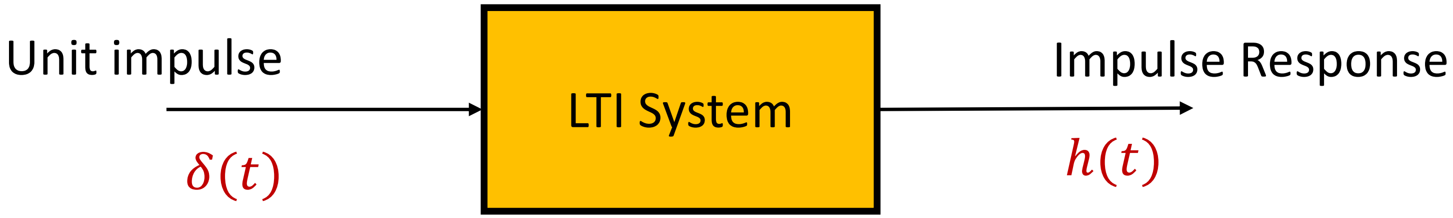
- For an integer t_o and arbitrary constants a_1 and a_2, LTI system property is

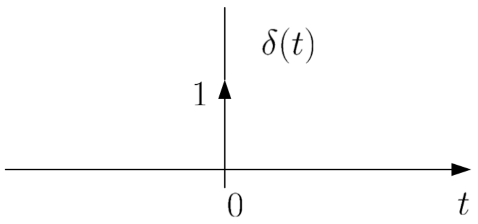
$$a_1x_1(t-t_o)+a_2x_2(t-t_o) \longrightarrow \boxed{H\{\cdot\}} \longrightarrow$$

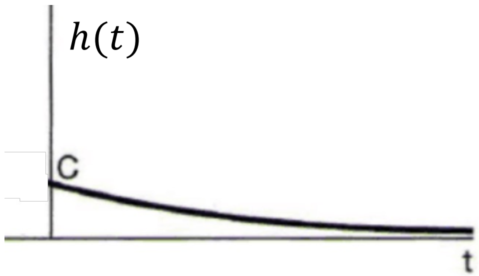
$$\begin{aligned} y(t) &= H\{a_1x_1(t-t_o)+a_2x_2(t-t_o)\} \\ &= a_1y_1(t-t_o)+a_2y_2(t-t_o) \end{aligned}$$

Impulse Response of Continuous-Time Systems

- If the input of a system is unit impulse signal $\delta(t)$, the corresponding output is called the impulse response $h(t)$ of the LTI system







Why Impulse Response?

- For any CT signal $x(t)$, its amplitude at $t = t_o$ can be represented as integral of product between the signal $x(t)$ and time-shifted impulse signal $\delta(t - t_o)$:

$$\int_{-\infty}^{\infty} x(t)\, \delta(t - t_o)\, dt = x(t - t_o)$$

- Then, we can represent any continuous-time signal as

$$\boxed{x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t - \tau)\, d\tau}$$

Why Impulse Response is so important?

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \longrightarrow \boxed{\begin{array}{c}\text{LTI System}\\ h(t)\end{array}} \longrightarrow y(t) = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$$

- Using the principle of time-invariance:

$$H\{\delta(t)\} = h(t) \quad => \quad H\{\delta(t-t_o)\} = h(t-t_o)$$

- Using the principle of linearity:

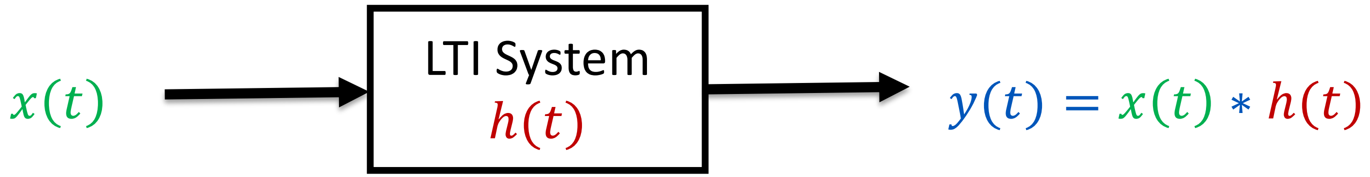
Convolution operator

$$y(t) = H\left\{\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau\right\} = \int_{-\infty}^{\infty} x(\tau)\,H\{\delta(t-\tau)\}\,d\tau = \int_{-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau = x(t) * h(t)$$

- **Therefore, any LTI system is completely described by its impulse response through the convolution operation.**

Convolution

- The output of any LTI system is a convolution operation of the input signal with the unit impulse response $h(t)$:

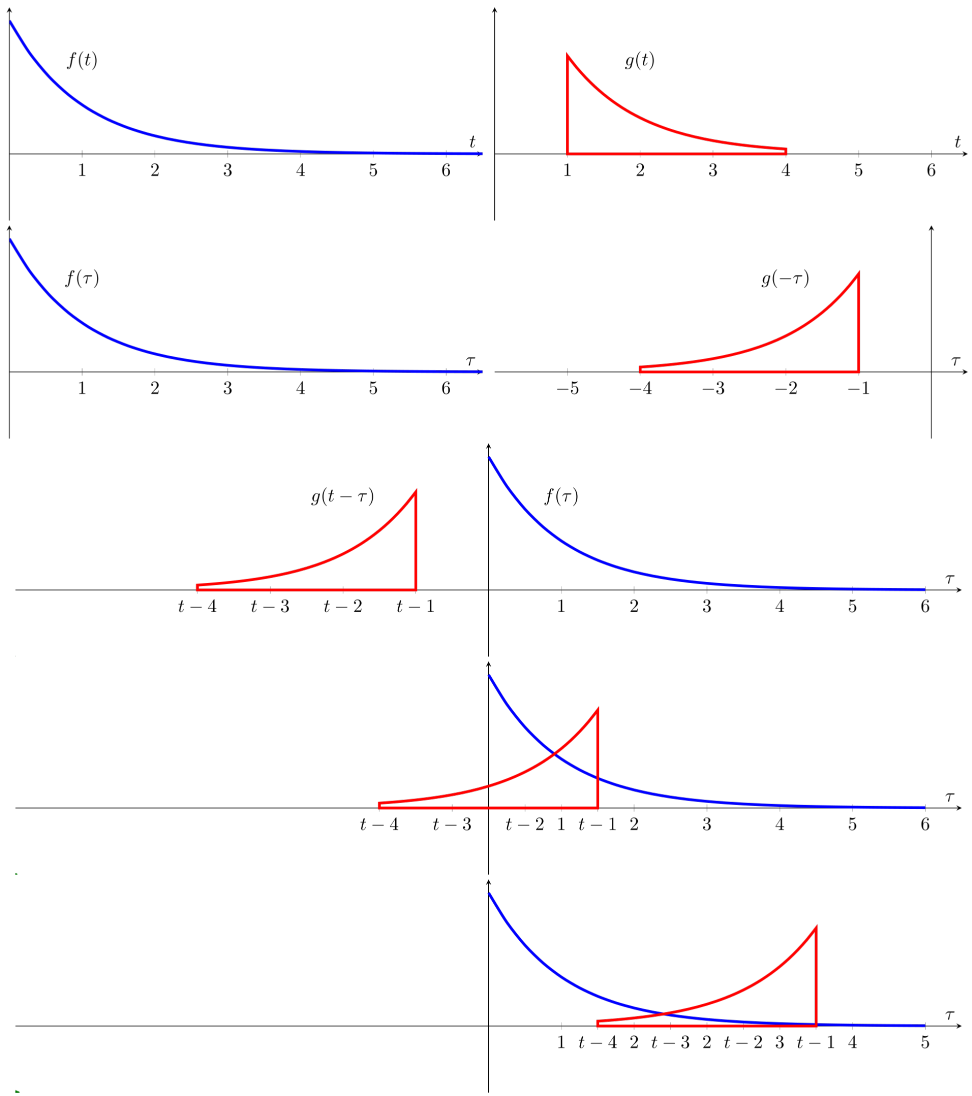


$$x(t) = \int_{-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau \quad \Rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau = x(t) * h(t)$$

- **Any Continuous-Time LTI system can be completely characterized by its unit impulse response $h(t)$**

Convolution Example

$$y(t) = f(t) * g(t)$$



Causality for LTI Systems

- A causal system only depends on present and past values of the input signal. We do not use knowledge about future information.
- For a continuous-time LTI system, convolution tells us that
  - $h(t) = 0 \quad for \;\; t < 0$
- It is because $y(t_o)$ must not depend on $x(t)$ for $t > t_o$, as the impulse response must be zero before the pulse!

$$y(t) = x(t) * h(t) = \int_{-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau$$

LTI System Stability

- Consider a bounded continuous-time input signal $x(t)$ with condition of:
  - $|x(t)| < M \quad for\ all\ t$
- Applying it to a LTI system with impulse response of $h(t)$ using convolution:
  - $y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$
- Using the triangle inequality (magnitude of a sum of a set of numbers is no larger than the sum of the magnitude of the numbers):
  - $|y(t)| \leq \int_{-\infty}^{\infty} |x(\tau)||h(t-\tau)|\, d\tau \leq M \int_{-\infty}^{\infty} |h(\tau)|\, d\tau$
- Therefore, a continuous-time **LTI system is BIBO stable** if and only if its impulse response $h(t)$ is absolutely integral, ie
  - $\int_{-\infty}^{\infty} |h(t)| dt \ < \infty$

Commutative Property

- Convolution is a commutative operator:

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(t)x(t-\tau)\, d\tau = \int_{-\infty}^{\infty} x(t)h(t-\tau)\, d\tau$$

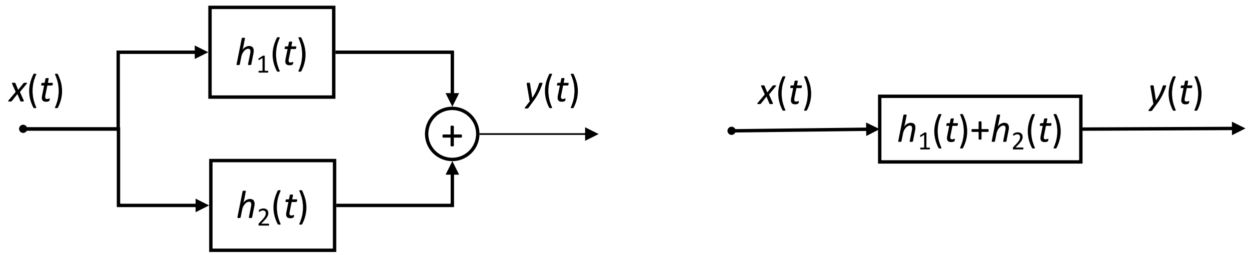
- Therefore, when calculating the response of a system to an input signal $x(t)$, we can imagine the signal being convolved with the unit impulse response $h(t)$, or vice versa, whichever appears the most straightforward.

Distributive Property (Parallel Systems)

- Another property of convolution is the distributive property

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$

- Therefore, the two systems are equivalent ◦



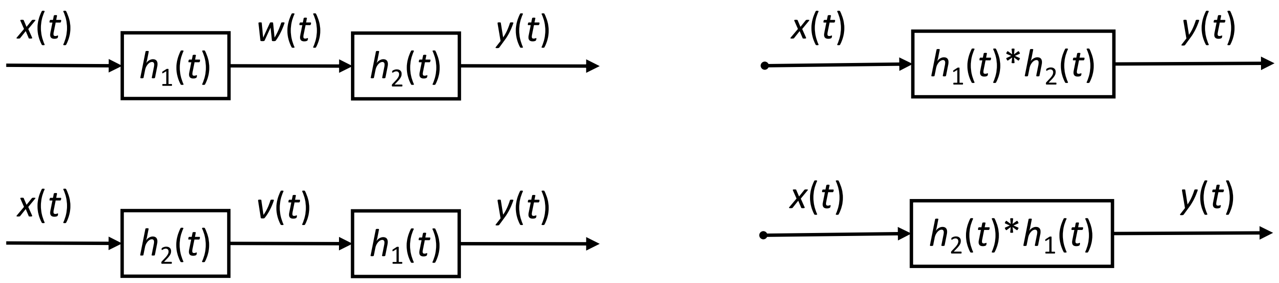
- The convolved sum of two impulse responses is equivalent to considering the two equivalent parallel system

Associative Property (Serial Systems)

- Another property of (LTI) convolution is that it is associative

$$x(t) * \big(h_1(t) * h_2(t)\big) = \big(x(t) * h_1(t)\big) * h_2(t)$$

- This can be easily verified by manipulating the integral indices
- Therefore, the following four systems are all equivalent and $y(t) = x(t) * h_1(t) * h_2(t)$ is unambiguously defined.



LTI System Memory

- An LTI system is memoryless if its output depends only on the input value at the same time, i.e.
  - $y(t) = k\,x(t)$
- For an impulse response, this can only be true if
  - $h(t) = k\,\delta(t)$
- This type of system is extremely simple, but the output of dynamic engineering, the physical system depends on the input value at the same time and the previous time.

On-line Math and Signal Processing Courses

- MIT - Highlight of Calculus by Gilbert Strang
- MIT - Linear Algebra by Gilbert Strang
- MIT - Differential Equations and Linear Algebra by Gilbert Strang and Cleve Moler
- MIT - Introduction to Probability by John Tsitsiklis and Patrick Jaillet
- MIT - Signals and Systems by Alan V. Oppenheim

Homework

- Try to form your term project group with 3 members
- Start to discuss the project direction
- Performing research on the selected topics

Start to Learn How to use Google Colab

- **Google Colab Tutorial**
  - Colab Tutorial
  - LaTeX and Markdown
- **Python Programming Tutorial**
  - **Socratica:** https://www.youtube.com/watch?v=bY6m6_IIN94&list=PLi01XoE8jYohWFPpC17Z-wWhPOSuh8Er-
  - **Sentdex**: https://www.youtube.com/watch?v=oVp1vrfL_w4&list=PLQVvvaa0QuDe8XSftW-RAxdo6OmaeL85M
- **Numpy Tutorials**
  - Learn Numpy - Array Indexing & Creation, Basic & Advanced Operations in 20 Minutes
  - Using NumPy Arrays to Perform Mathematical Operations in Python
  - NumPy for MATLAB users
  - NumPy Tutorial | SciPy 2020 | Eric Olsen
- **DSP with Python**
  - Youtub Tutorial : DSP using Python and MATLAB
  - Allen Downey - Introduction to Digital Signal Processing - PyCon 2018
  - Python Audio
  - Digital signal processing through speech, hearing, and Python