z-Transform Properties and LTI System Analysis

EE4015 Digital Signal Processing

Dr. Lai-Man Po

Department of Electrical Engineering

City University of Hong Kong

Overview

- Properties of z-transform
- Transfer Function
- Transfer Function & Difference Equation
- Transfer Function & Impulse Response
- Transfer Function & System Stability
- Difference Equation & System Stability
- Pole-Zero Plot
- Stability Analysis based Pole-Zero Plot

Key Properties of the z-Transform

1. Linearity : $x_1[n] \leftrightarrow X_1(z)\ ROC = R_1$ and $x_2[n] \leftrightarrow X_2(z)\ ROC = R_2$

   $$ax_1[n] + bx_2[n] \leftrightarrow aX_1(z) + bX_2(z) \quad ROC = R_1 \cap R_2$$

2. Time Shifting : $x[n - n_o] \leftrightarrow z^{-n_o}X(z)$
3. Time Reversal : $x[-n] \leftrightarrow X(1/z)$
4. Exponential Scaling : $a^n x[n] \leftrightarrow X(z/a)$
5. Z-domain Differentiation : $nx[n] \leftrightarrow z\dfrac{dX(x)}{dz}$
6. Convolution : $x_1[n] * x_2[n] \leftrightarrow X_1(z)X_2(z) \quad ROC = R_1 \cap R_2$

Linearity of the z-Transform

- If $x_1[n] \leftrightarrow X_1(z)$ for z in ROC of R_1 and $x_2[n] \leftrightarrow X_2(z)$ for z in ROC of R_2 then

$$ax_1[n] + bx_1[n] \leftrightarrow aX_1(z) + bX_2(z) \text{ for } z \text{ in ROC of } R_1 \cap R_2$$

- Let $y[n] = ax_1[n] + bx_2[n]$ then

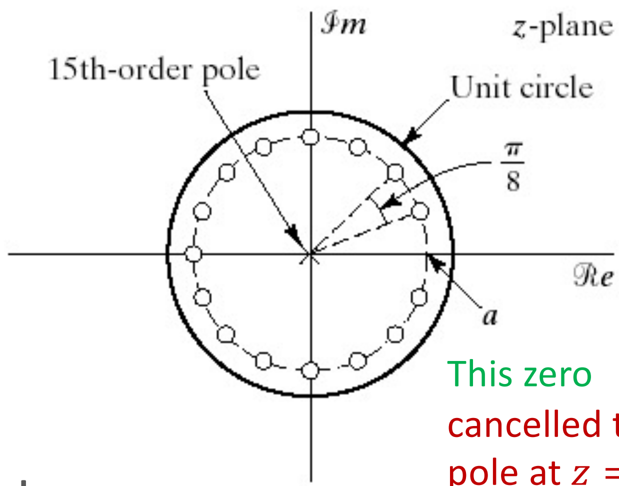
$$Y(z) = \sum_{n=-\infty}^{+\infty} y[n]z^{-n} = \sum_{n=-\infty}^{+\infty} \left(ax_1[n] + bx_2[n]\right)z^{-n}$$

$$= a\sum_{n=-\infty}^{+\infty} x_1[n]z^{-n} + b\sum_{n=-\infty}^{+\infty} x_2[n]z^{-n} = aX_1(z) + bX_2(\mathrm{z})$$

$$ROC = R_1 \cap R_2$$

ROC Property of the z-Transform

- Note that the ROC of combined sequence may be larger than either ROC
  - This would happen if some pole/zero cancellation occurs
- Example: $x[n] = a^n u[n] - a^n u[n-N]$
  - Both sequences are right-sided
  - Both sequences have a pole $z = a$
  - Both have a ROC defined as $|z| > |a|$
  - In the combined sequence the pole at $z = a$ cancels with a zero at $z = a$
  - The combined ROC is the entire z plane except z=0



This zero cancelled the pole at $z = a$

ROC: $|z| > 0$

$$X(z) = \frac{1}{z^{N-1}} \cdot \frac{z^N - a^N}{z - a}$$

Linearity Example 1

Find the z-transform of the signal $x[n]$ defined by

$$x[n] = u[n] - (0.5)^n u[n]$$

- Applying the Linear property of the z-transform, we have

$$X(z) = Z\{x[n]\} = Z\{u[n] - (0.5)^n u[n]\}$$

$$= Z\{u[n]\} - Z\{-(0.5)^n u[n]\}$$

$$= \frac{z}{z-1} - \frac{z}{z-0.5}, \quad ROC = |\mathrm{z}| > 1$$

Linearity Example 2

Find the z-transform of the signal $x[n]$ defined by

$$x[n] = [3(2)^n - 4(3)^n]u[n]$$

- Applying the linearity of the z-transform, we have
  - As we know $Z\{(a)^n u[n]\} = \dfrac{1}{1 - az^{-1}}$ $\quad ROC: |z| > |a|$
  - Applying the linearity of the z-transform, we have

$$X(z) = 3\frac{1}{1 - 2z^{-1}} - 4\frac{1}{1 - 3z^{-1}} \qquad ROC: |z| > 2 \cap |z| > 3$$

$$= 3\frac{z}{z - 2} - 4\frac{z}{z - 3} \qquad ROC: |z| > 3$$

Linearity Example 3

Find the z-transform of the signal $x[n]$ defined by $x[n] = \cos(\omega_o n)u[n]$

- We know

$$\cos(\omega_o n) = \frac{e^{j\omega_o n} + e^{-j\omega_o n}}{2} = \frac{1}{2}e^{j\omega_o n} + \frac{1}{2}e^{-j\omega_o n}$$

$$Z\{e^{j\omega_o n}u[n]\} = \frac{1}{1 - e^{j\omega_o}z^{-1}}, \quad ROC = |z| > 1$$

- Applying the linearity of the z-transform, we have

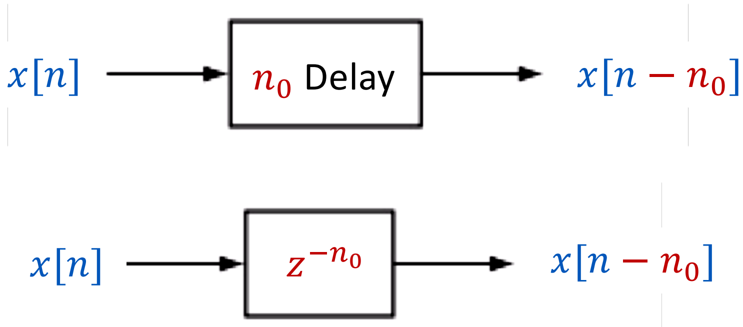
$$X(z) = \frac{1}{2}Z\{e^{j\omega_o n}u[n]\} + \frac{1}{2}Z\{e^{-j\omega_o n}u[n]\} = \frac{1}{2}\left[\frac{1}{1 - e^{j\omega_o}z^{-1}}\right] + \frac{1}{2}\left[\frac{1}{1 - e^{-j\omega_o}z^{-1}}\right]$$

$$= \frac{1}{2}\left[\frac{1 - e^{-j\omega_o}z^{-1}}{(1 - e^{j\omega_o}z^{-1})(1 - e^{-j\omega_o}z^{-1})}\right] + \frac{1}{2}\left[\frac{1 - e^{j\omega_o}z^{-1}}{(1 - e^{-j\omega_o}z^{-1})(1 - e^{j\omega_o}z^{-1})}\right] = \frac{1}{2}\left[\frac{1 - e^{-j\omega_o}z^{-1} + 1 - e^{j\omega_o}z^{-1}}{(1 - e^{-j\omega_o}z^{-1})(1 - e^{j\omega_o}z^{-1})}\right]$$

$$= \frac{1}{2}\left[\frac{2 - z^{-1}(e^{j\omega_o} + e^{-j\omega_o})}{1 - z^{-1}(e^{j\omega_o} + e^{-j\omega_o}) + z^{-2}}\right] = \frac{1 - z^{-1}\cos(\omega_o)}{1 - 2z^{-1}\cos(\omega_o) + z^{-2}} \qquad ROC = |z| > 1$$

Time Shifting of the z-Transform

- A n_0-sample delay in the time domain appears in the z domain as a z^{-n_0} factor.
- More generally, $X(z) = \sum_{n=-\infty}^{+\infty} x[n]z^{-n}$



- Let $y[n] = x[n-n_0]$

$$Y(z) = \sum_{n=-\infty}^{+\infty} y[n]z^{-n} = \sum_{n=-\infty}^{+\infty} x[n-n_0]z^{-n}$$

- Substitute $m = n - n_0$

$$Y(z) = \sum_{m=-\infty}^{+\infty} x[m]z^{-m-n_0} = z^{-n_0}\sum_{m=-\infty}^{+\infty} x[m]z^{-m} = z^{-n_0}X(z)$$

Time Shifting Property Example

Find the z-transform of the signal $x[n]$ using time shifting property

$$x[n] = (0.5)^{n-5}u[n-5]$$

Solution

$$Z\{(0.5)^n u[n]\} = \frac{1}{1-0.5z^{-1}}, \qquad ROC: |z| > 0.5$$

- Applying the time shifting property of the z-transform, we have

$$X(z) = z^{-5} Z\{(0.5)^n u[n]\}$$

$$= z^{-5}\frac{1}{1-0.5z^{-1}} = \frac{z^{-4}}{z-0.5} \qquad ROC = |z| > 0.5$$

Time Reversal Property

- $x[-n] \leftrightarrow X(1/z) \qquad ROC = 1/R_x$

Proof

$$\mathcal{Z}\{x[-n]\} = \sum_{n=-\infty}^{+\infty} x[-n]z^{-n} = \underbrace{\sum_{m=\infty}^{-\infty}}^{\boxed{m=-n}} x[m]z^{m} = \sum_{m=-\infty}^{+\infty} x[m](z^{-1})^{-m}$$

$$= X(z^{-1})$$

Time Reversal Property Example

Find the z-transform of the signal $x[n] = a^{-n}u[-n]$

Solution

- The sequence of $a^{-n}u[-n]$ is Time-reversed version of $a^{n}u[n]$
- Applying the time reversal theorem of the z-transform, we have

$$X(z) = \frac{1}{1-az} = \left(\frac{1}{-a}\right)\frac{1}{z-\frac{1}{a}} = \frac{-a^{-1}z^{-1}}{1-a^{-1}z^{-1}}, \quad |z| < |a^{-1}| \qquad \text{ROC is inverted}$$

Exponential Scaling Property

$$a^n x[n] \overset{Z}{\leftrightarrow} X(z/a) \qquad ROC = |a|R_x$$

- ROC is scaled by $|a|$
- All pole/zero locations are scaled
- If a is a positive real number: z-plane shrinks or expands
- If a is a complex number with unit magnitude it rotates

Exponential Scaling Example

- Example: We know the z-transform pair

$$u[n] \overset{Z}{\leftrightarrow} \frac{1}{1-z^{-1}} \qquad ROC = |z| > 1$$

- Let's find the z-transform of

$$x[n] = r^n \cos(\omega_o n)\, u[n] = \frac{1}{2}\left(re^{j\omega_o}\right)^n u[n] + \frac{1}{2}\left(re^{-j\omega_o}\right)^n u[n]$$

$$X(z) = \frac{1/2}{1 - re^{j\omega_o} z^{-1}} + \frac{1/2}{1 - re^{-j\omega_o} z^{-1}} \qquad ROC = |z| > r$$

Z-Domain Differentiation Property

$$nx[n] \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz} \qquad ROC = R_x$$

- For example, we want the inverse z-transform of

$$X(z) = \log(1 + az^{-1}) \qquad |z| > |a|$$

- Let's differentiate to obtain rational expression

$$\frac{dX(z)}{dz} = -\frac{-az^{-2}}{1 + az^{-1}} \Rightarrow -z\frac{dX(z)}{dz} = az^{-1}\frac{1}{1 + az^{-1}}$$

- Making use of z-transform properties and ROC

$$nx[n] = a(-a)^{n-1}u[n-1]$$

$$x[n] = (-1)^{n-1}\frac{a^n}{n}u[n-1]$$

The z-Transform Convolution Property

- Convolution in time domain is equal to the multiplication in frequency domain and vice versa.
- If $x[n] \leftrightarrow X(z)$ and $y[n] \leftrightarrow Y(z)$, then

$$x[n] * y[n] \leftrightarrow X(z)Y(z) \qquad ROC = R_{\mathrm{x}} \cap R_{\mathrm{y}}$$

- Proof

$$Z\{x[n] * y[n]\} = \sum_{n=-\infty}^{\infty} \left\{ \sum_{k=-\infty}^{\infty} x[k]y[n-k] \right\} z^{-n}$$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{n=-\infty}^{\infty} y[n-k]z^{-n} = \sum_{k=-\infty}^{\infty} x[k]z^{-k}Y(z) = X(z)Y(z)$$

Convolution Property Example 1

- Consider the two sequences

$$x_1[n] = 3\delta[n] + 2\delta(n-1) \qquad\qquad x_2[n] = 2\delta[n] - \delta(n-1)$$

- Find the z-transform of convolution

$$x[n] = x_1[n] * x_2[n]$$

- Determine the convolution sum using the z-transform

Solution

$$X_1(z) = \mathrm{Z}\{x_1[n]\} = 3 + 2\mathrm{z}^{-1}$$

$$X_2(z) = \mathrm{Z}\{x_2[n]\} = 2 - \mathrm{z}^{-1}$$

$$X(z) = \mathrm{Z}\{x_1[n] * x_2[n]\} = X_1(z)X_2(z) = 6 + \mathrm{z}^{-1} - 2\mathrm{z}^{-2}$$

$$x[n] = Z^{-1}\{X(z)\} = 6\delta[n] + \delta(n-1) - 2\delta(n-2)$$

Convolution Property Example 2

- Compute the convolution of the following two sequences using z-transform

$$x_1[n] = [1, \quad -2, \quad 1] \qquad x_2[n] = \begin{cases} 1, & 0 \le n < 5 \\ 0, & elsewhere \end{cases} = [1, 1, 1, 1, 1]$$

Solution

$$X_1(z) = \mathrm{Z}\{x_1[n]\} = 1 - 2\mathrm{z}^{-1} + z^{-2}$$

$$X_2(z) = \mathrm{Z}\{x_2[n]\} = 1 + \mathrm{z}^{-1} + z^{-2} + \mathrm{z}^{-3} + z^{-4}$$

$$X(z) = X_1(z)X_2(z) = 1 - \mathrm{z}^{-1} - z^{-5} + \mathrm{z}^{-6}$$

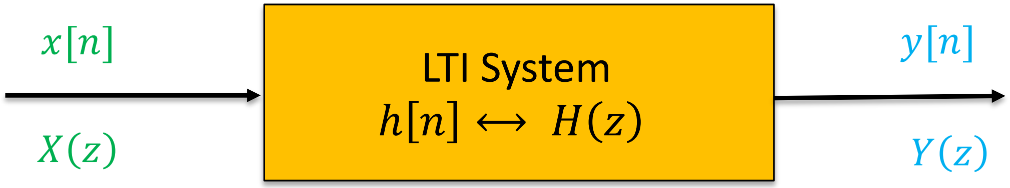
$$x[n] = x_1[x] * x_2[n] = Z^{-1}\{X(z)\} = [1, -1, 0, 0, 0, -1, 1]$$

Properties of the z-Transform

Property	Time Domain	z-Domain	ROC
Notation	$x(n)$ $x_1(n)$ $x_2(n)$	$X(z)$ $X_1(z)$ $X_2(z)$	ROC: $r_2 < \|z\| < r_1$ ROC_1 ROC_2
Linearity	$a_1x_1(n) + a_2x_2(n)$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2
Time shifting	$x(n-k)$	$z^{-k}X(z)$	That of $X(z)$, except $z = 0$ if $k > 0$ and $z = \infty$ if $k < 0$
Scaling in the z-domain	$a^n x(n)$	$X(a^{-1}z)$	$\|a\|r_2 < \|z\| < \|a\|r_1$
Time reversal	$x(-n)$	$X(z^{-1})$	$\frac{1}{r_1} < \|z\| < \frac{1}{r_2}$
Conjugation	$x^*(n)$	$X^*(z^*)$	ROC
Real part	$\text{Re}\{x(n)\}$	$\frac{1}{2}[X(z) + X^*(z^*)]$	Includes ROC
Imaginary part	$\text{Im}\{x(n)\}$	$\frac{1}{2}[X(z) - X^*(z^*)]$	Includes ROC
Differentiation in the z-domain	$nx(n)$	$-z\frac{dX(z)}{dz}$	$r_2 < \|z\| < r_1$
Convolution	$x_1(n) * x_2(n)$	$X_1(z)X_2(z)$	At least, the intersection of ROC_1 and ROC_2
Correlation	$r_{x_1x_2}(l) = x_1(l) * x_2(-l)$	$R_{x_1x_2}(z) = X_1(z)X_2(z^{-1})$	At least, the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$
Initial value theorem	If $x(n)$ causal	$x(0) = \lim_{z\to\infty} X(z)$	
Multiplication	$x_1(n)x_2(n)$	$\frac{1}{2\pi j}\oint_C X_1(v)X_2\left(\frac{z}{v}\right)v^{-1}dv$	At least $r_{1l}r_{2l} < \|z\| < r_{1u}r_{2u}$
Parseval's relation	$\sum_{n=-\infty}^{\infty} x_1(n)x_2^*(n) = \frac{1}{2\pi j}\oint_C X_1(v)X_2^*(1/v^*)v^{-1}dv$		

LTI System Analysis Using the z-Transform

Transfer Function

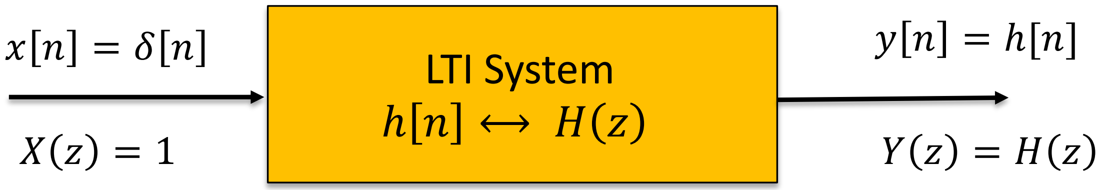


$H(z) = \dfrac{Y(z)}{X(z)}$ is referred as Transfer Function of the system.

- It is the ratio of the output to the input in the z domain:
  - $Y(z)$ is the z transform of the output $y[n]$
  - $X(z)$ is the z-transform of the input $y[n]$
  - $H(z)$ is the z-transform of the **impulse response** $h[n]$

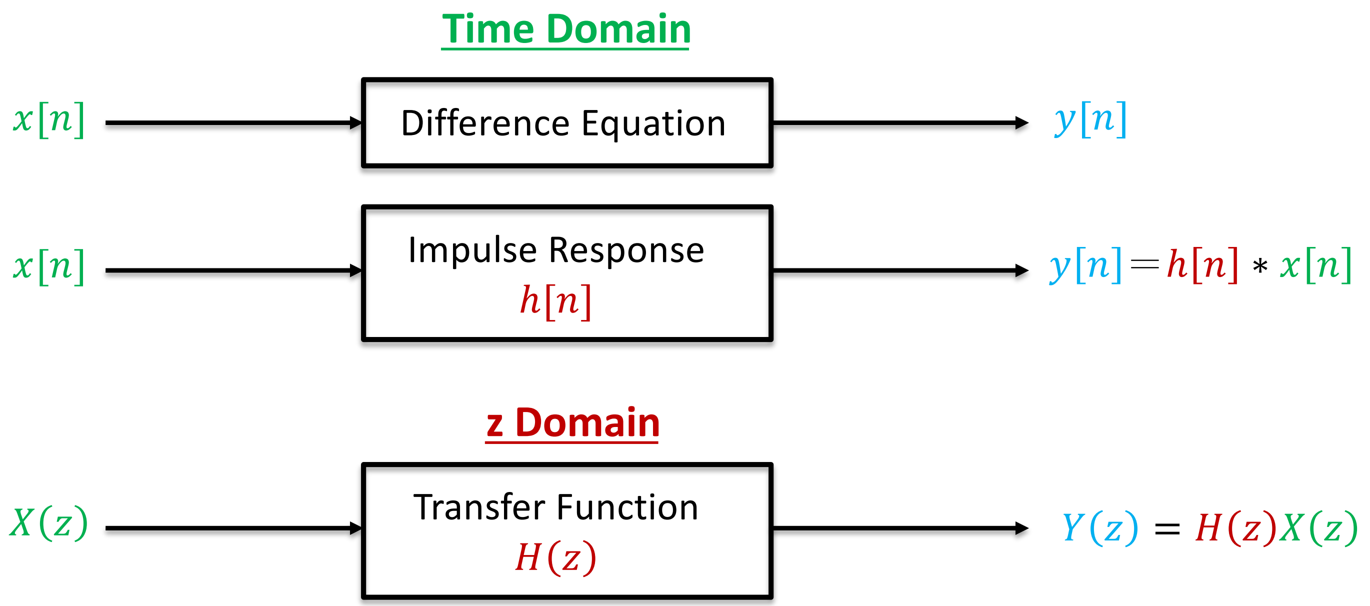
Impulse Response

- The impulse response $h[n]$ of the discrete-time LTI system $H(z)$ can be obtained by solving its difference equation using a unit impulse input $\delta[n]$
- With the help of the z-transform and noticing that $X(z) = Z\{\delta[n]\} = 1$
  - $h[n] = Z^{-1}\{Y(z)/X(z)\} = Z^{-1}\{H(z)\}$



System Outputs in Time and z domains

- The LTI system output can be find using three different ways.



Rational Transfer Function

- Transfer function can be expressed as a **rational function** consist of numerator polynomial divided by denominator polynomial.

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} + a_N z^{-N}} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{\sum_{k=0}^{N} a_k z^{-k}}$$

- The highest power in a polynomial is called its degree.
- In a **proper rational function**, the degree of the numerator is less than or equal to the degree of the denominator. ($M \leq N$)
- In a **strictly proper rational function**, the degree of the numerator is less than the degree of the denominator. ($M < N$)
- In an **improper rational function**, the degree of the numerator is greater than the degree of the denominator. ($M > N$)

Transfer Function and Difference Equation

- A linear constant coefficient difference equation be described by a rational function in z-transform as a **ratio of Polynomials** in z.

$$a_0 y[n] + a_1 y[n-1] + \cdots + a_M y[n-N] = b_0 x[n] + b_1 x[n-1] + \cdots + b_N x[n-M]$$

- **Taking the z-transform of both sides**

$$a_0 Y(z) + a_1 z^{-1} Y(z) + \cdots + a_N z^{-N} Y(z) = b_0 X(z) + b_1 z^{-1} X(z) + \cdots + b_M z^{-M} X(z)$$

$$Y(z)(a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}) = X(z)(b_0 + b_1 z^{-1} + \cdots + b_M z^{-M})$$

- Taking $Y(z)$ and $X(z)$ common and then cross multiply to get Transfer Function $H(z)$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} + a_N z^{-N}}$$

Difference Equation Example 1

Find the transfer function described by the following difference equation.

$$2y[n] + y[n-1] + 0.9y[n-2] = x[n-1] + x[n-4]$$

Solution: Taking the z-transforms term by term we get,

$$2Y(z) + z^{-1}Y(z) + 0.9z^{-2}Y(z) = z^{-1}X(z) + z^{-4}X(z)$$

Factoring out $Y(z)$ on the left side and $X(z)$ on the right side:

$$Y(z)(2 + z^{-1} + 0.9z^{-2}) = X(z)(z^{-1} + z^{-4})$$

The transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^{-1} + z^{-4}}{2 + z^{-1} + 0.9z^{-2}}$$

Difference Equation Example 2

Find the transfer function described by the following difference equation.

$$y[n] - 0.2y[n-1] = x[n] + 0.8x[n-1]$$

Solution: Taking z transforms term by term we get,

$$Y(z) - 0.2z^{-1}Y(z) = X(z) + 0.8z^{-1}X(z)$$

Factoring out $Y(z)$ on the left side and $X(z)$ on the right side:

$$Y(z)(1 - 0.2z^{-1}) = X(z)(1 + 0.8z^{-1})$$

The transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 + 0.8z^{-1}}{1 - 0.2z^{-1}}$$

Difference Equation Example 3

Find the transfer function described by the following difference equation.

$$y[n] = 0.75x[n] - 0.3x[n-2] + 0.01x[n-3]$$

Solution: Taking z transforms term by term we get,

$$Y(z) = 0.75X(z) - 0.3z^{-2}X(z) + 0.01z^{-3}X(z)$$

Factoring out $Y(z)$ on the left side and $X(z)$ on the right side:

$$Y(z) = X(z)(0.75 - 0.3z^{-2} + 0.01z^{-3})$$

The transfer function is

$$H(z) = \frac{Y(z)}{X(z)} = 0.75 - 0.3Z^{-2} - 0.01Z^{-3}$$

Difference Equation Example 4

Find the difference equation that correspond to transfer function.

$$H(z) = \frac{1 \; + \; 0.5z^{-1}}{1 \; - \; 0.5z^{-1}}$$

Solution: Since $H(z) = Y(z)/X(z)$, do the cross multiply to get

$$Y(z)(1 - 0.5z^{-1}) = X(z)(1 + 0.5z^{-1})$$

then $$Y(z) - 0.5z^{-1}Y(z) = X(z) + 0.5z^{-1}X(z)$$

Finally taking the inverse z-transform term by term to get

$$y[n] - 0.5y[n-1] = x[n] + 0.5x[n-1]$$

$$\Rightarrow \quad y[n] = x[n] + 0.5x[n-1] + 0.5y[n-1]$$

Difference Equation Example 5

Find the difference equation that correspond to transfer function.

$$H(z) = \frac{1 \ + \ 0.8z^{-1}}{1 \ - \ 0.2z^{-1} + 0.7z^{-2}}$$

Solution: Since $H(z) = Y(z)/X(z)$, do the cross multiply to get

$$Y(z)(1 - 0.2z^{-1} + 0.7z^{-2}) = X(z)(1 + 0.8z^{-1})$$

then $$Y(z) - 0.2z^{-1}Y(z) + 0.7z^{-2}Y(z) = X(z) + 0.8z^{-1}X(z)$$

Finally taking the inverse z-transform term by term to get

$$y[n] - 0.2y[n-1] + 0.7y[n-2] = x[n] + 0.8x[n-1]$$

$$\Rightarrow \quad y[n] = x[n] + 0.8x[n-1] + 0.2y[n-1] - 0.7y[n-2]$$

Difference Equation Example 6

Find the difference equation that correspond to transfer function.

$$H(z) = \frac{z}{(2z-1)(4z-1)}$$

Solution: $H(z) = \frac{z}{8z^2-6z+1} = \frac{Y(z)}{X(z)}$

Do the cross multiply to get

$(8z^2 - 6z + 1)Y(z) = (z)X(z)$, then $8z^2Y(z) - 6zY(z) + Y(z) = zX(z)$

$$\Rightarrow \quad 8Y(z) - 6z^{-1}Y(z) + z^{-2}Y(z) = z^{-1}X(z)$$

Finally taking the inverse z-transform term by term to get

$$8y[n] - 6y[n-1] + y[n-2] = x[n-1]$$

$$\Rightarrow \quad y[n] - 0.75y[n-1] + 0.125y[n-2] = 0.125x[n-1]$$

$$\Rightarrow \quad y[n] = 0.125x[n-1] + 0.75y[n-1] - 0.125y[n-2]$$

Pole-Zero Description of Discrete-Time System

- The **zeros** of a z-transform $H(z)$ are the values of *z* for which $H(z) = 0$.
- The **poles** of a z-transform are the values of *z* for which $H(z) = \infty$.
- If *H(z)* is a rational function , then

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} + a_N z^{-N}}$$

- After factoring the rational transfer function, the roots β_k of the numerator polynomial are zeros and roots α_k of denominator polynomial are poles.

$$H(z) = \frac{K(z-\beta_1)(z-\beta_2)\cdots(z-\beta_M)}{(z-\alpha_1)(z-\alpha_1)\cdots(z-\alpha_N)} = \frac{K\prod_{k=1}^{M}\left(1-\beta_k z^{-k}\right)}{\prod_{k=1}^{N}\left(1-\alpha_k z^{-k}\right)}$$

The poles and zeros of a system can provide a great deal of information about the behavior of the system.

Identify Poles and Zeros (1)

- It is easiest to identify the poles and zeros if the rational transfer function

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_{M-1} z^{M-1} + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_{N-1} z^{N-1} + a_N z^{-N}}$$

is converted to the form

$$H(z) = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

which has **only positive exponents**.

Identify Poles and Zeros (2)

$$H(z) = z^{N-M} \frac{b_0 z^M + b_1 z^{M-1} + \cdots + b_M}{a_0 z^N + a_1 z^{N-1} + \cdots + a_N}$$

- The roots of the numerator polynomial are the zeros of the system.
- The roots of the denominator polynomial are the poles of the system.
- In general, numerator and denominator polynomials can always be factored

$$H(z) = K \frac{(z-\beta_1)(z-\beta_2)\cdots(z-\beta_M)}{(z-\alpha_1)(z-\alpha_2)\cdots(z-\alpha_N)} = K \frac{\prod_{k=1}^{M}\left(1-\beta_k z^{-k}\right)}{\prod_{k=1}^{N}\left(1-\alpha_k z^{-k}\right)}$$

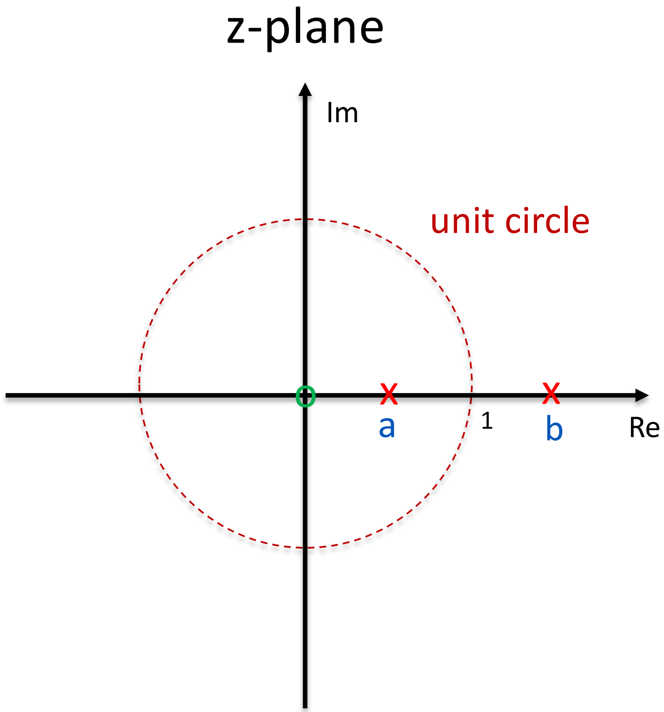
Where the are β_k the zeros, are α_k the poles, and K is called the gain

Effects of Poles and Zeros

- Poles are the values of z that make the denominator of a transfer function zero.
  - Poles have the biggest effect on the behavior of discrete-time LTI system
- Zeros are the values of z that make the numerator of a transfer function zero.
  - Zeros tend to modulate, to a greater or lesser degree depending on their position relative to the poles.
- The poles of the system can be found if its transfer function is known.
- Both zeros and poles are in general complex numbers.

Pole-Zero Plot

- A very powerful tool for the discrete-time system analysis and design is a complex plane called z-plane, on which poles and zeros of the transfer function are plotted.
- On the z plane,
  - poles are plotted as crosses (X)
  - zeros are plotted as circles (O)
- A plot showing pole and zero locations is called a pole-zero plot.



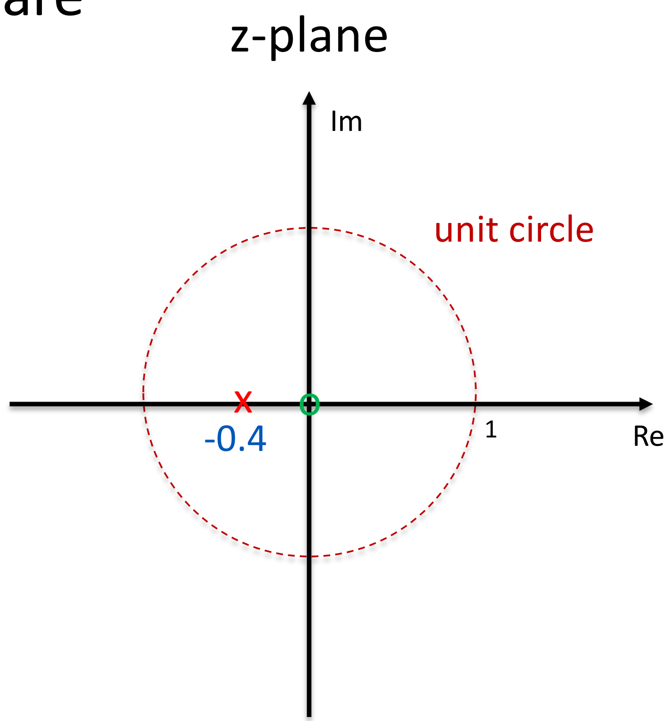
Pole-Zero Plot

Pole-Zero Plot Example 1

For a first order system the poles and zeros are

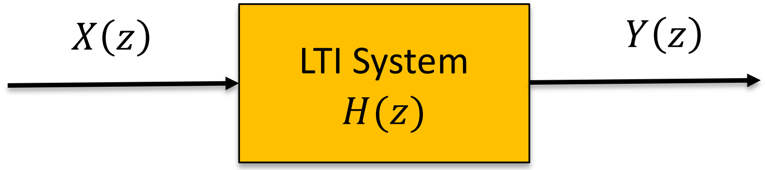
$$H(z) = \frac{2}{1 + 0.4z^{-1}}$$

- One Pole at z = -0.4
- One Zero at z = 0



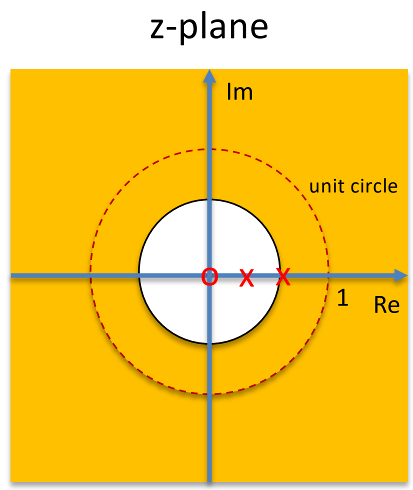
LTI System Analysis using the z-Transform

- $y[n] = x[n] * h[n]$



- $Y(z) = X(z)\, H(z)$
- $H(z)$ is the z-transform of the impulse response $h[n]$, which is called transfer function
- **Stable system** <=> Unit circle in the ROC
- **Causal system** => $h[n]$ is right-sided sequence

  => ROC outside outermost pole



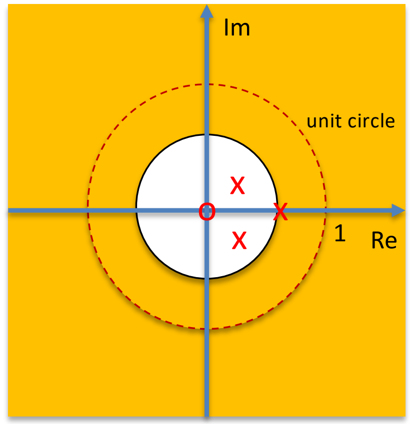
System Stability based Pole Locations

- The position of the poles and zeros on the z-plane can give clue about the way a discrete-time system will behave.
- One reason the poles of a system are so useful is that they determine whether the system is stable or not.
- The system is stable if the poles lie inside the unit circle, which is a circle of unit radius on the z-plane.
- Since poles are complex numbers, this requires that their magnitudes be less than one.
- Mathematically, the region of stability can be described as $|z| < 1$

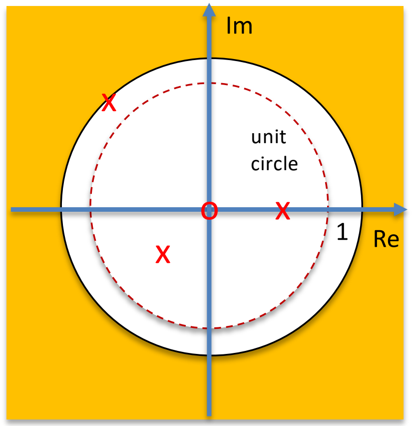
Stable Causal LTI System

- If the magnitude of each pole is less than one, the poles are less than one unit's distance from the center of the unit circle, and the system is ***stable***. The ROC includes unit circle.
- If any of the poles of a system lie outside the unit circle, the system is ***unstable***.
- If the outermost pole lies on the unit circle, the filter is described as being ***marginally stable***

z-plane



Stable System

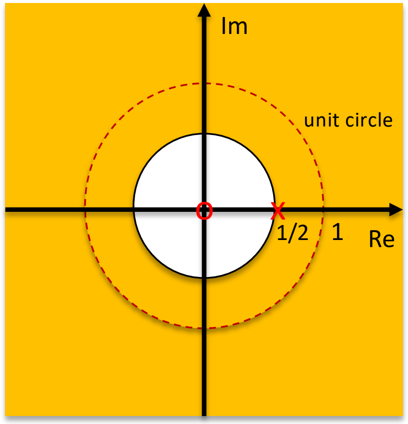


Unstable System

ROC of $H(z) = \frac{1}{1-\frac{1}{2}z^{-1}}$

ROC: |z|>1/2

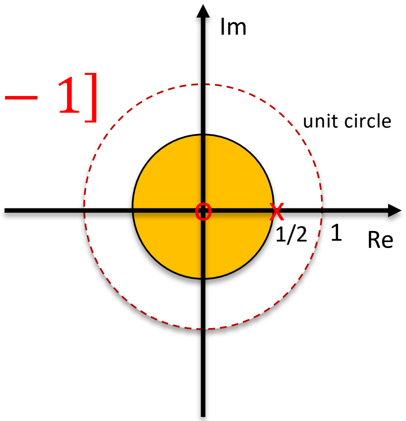
- Stable system (unit circle in ROC)
- Casual system (h[n] is right-sided sequence)
- $h[n] = \left(\frac{1}{2}\right)^n u[n]$



z-plane

ROC: |z|<1/2

- Unstable system (unit circle not in ROC)
- Non-casual system (h[n] is left-sided sequence)
- $h[n] = -\left(\frac{1}{2}\right)^n u[-n-1]$



z-plane

System Stability Example 1

Find the poles and zeros and stability for the causal discrete-time system whose transfer function is

$$H(z) = \frac{4z^{-1}}{4 - 9z^{-1} + 2z^{-2}}$$

Solution

Eliminating negative exponents yields

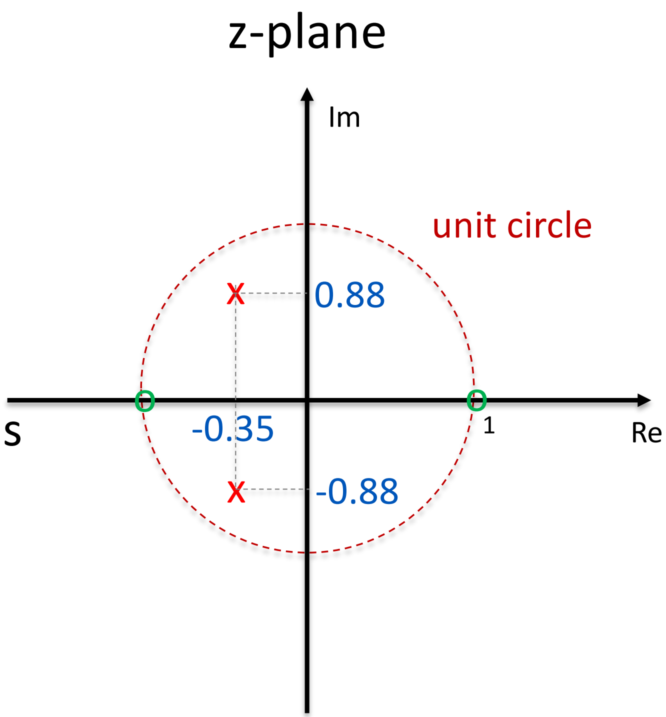
$$H(z) = \frac{4z^{-1}}{4-9z^{-1}+2z^{-2}} = \frac{z^{-1}}{1-2.25z^{-1}+0.5z^{-2}} = \frac{z^{-1}}{(1-0.25z^{1})(1-2z^{-1})} \cdot \frac{z^2}{z^2} = \frac{z}{(z-0.25)(z-2)}$$

- Two Poles at z = 0.25 and z = 2
- One Zeros at z = 0
- As one pole lie outside the unit circle at z = 2, hence the system is ***unstable***.

System Stability Example 2

Find the poles and zeros and stability for the causal discrete-time system whose transfer function is

$$H(z) = \frac{1 - z^{-2}}{1 + 0.7z^{-1} + 0.9z^{-2}}$$



Solution

Two zeros at 0.

Two poles are located at $-0.35 \pm j0.88$

- For these poles the distance from the center of the unit circle is
  - $|z| = \sqrt{(-0.35)^2 + (0.88)^2} = 0.9487 < 1$

As both poles lie inside the unit circle,

Therefore, the system is ***stable***.

System Stability Example 3

Determine the stability of the following causal system.

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}}$$

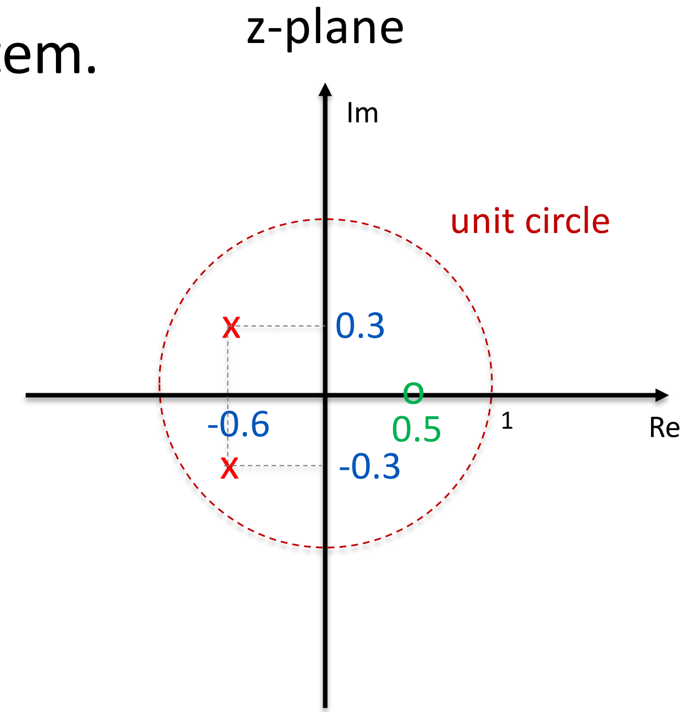
Solution: Eliminating negative exponents yields

$$H(z) = \frac{z^{-1} - 0.5z^{-2}}{1 + 1.2z^{-1} + 0.45z^{-2}} \cdot \frac{z^2}{z^2} = \frac{z - 0.5}{z^2 + 1.2z + 0.45}$$

Poles: at $z = -0.6 + j0.3$ and $z = -0.6 - j0.3$

Zero: at $z = 0.5$

As all poles lie inside the unit circle, hence the system is ***stable***.



System Stability Example 4

Find the stability of the filter if the difference equation of the filter is

$$y[n] + 0.8y[n-1] - 0.9y[n-2] = x[n-2]$$

Solution

- Poles are found most easily from the transfer function.

$$H(z) = \frac{z^{-2}}{1+0.8z^{-1}-0.9z^{-2}} \cdot \frac{z^2}{z^2} = \frac{1}{z^2+0.8z-0.9}$$

- The quadratic formula gives the pole locations as

$$z = \frac{-0.8 \pm \sqrt{0.8^2 - 4(1)(-0.9)}}{2(1)} = \frac{-0.8 \pm 2.059}{2} = 0.630 \quad and \quad -1.430$$

- The poles in this case are purely real, without any imaginary component. Clearly the pole at $z = -1.430$ lies outside the unit circle, so the system is **unstable**.

Example 5: LTI System Analysis in z Domain

- Given the following system function:

$$H(z) = \frac{1+0.25z^{-1}}{1+0.8z^{-1}-0.84z^{-2}}$$

(a) Plot the pole-zero diagram of $H(z)$.

(b) Find a stable impulse response $h[n]$.

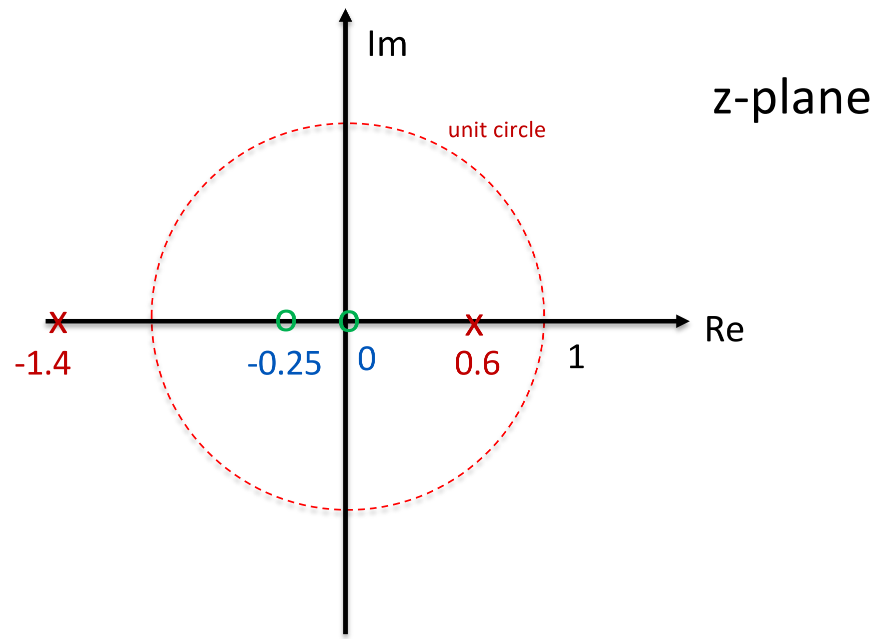
(c) Find a causal impulse response exist that is both stable and causal?

Solution (a)

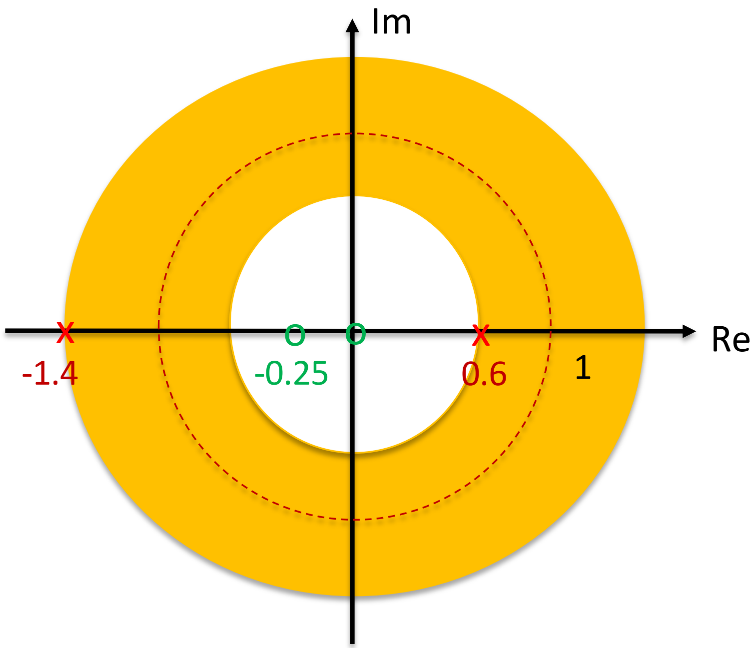
Plot the **pole-zero diagram** of $H(z)$.

$$H(z) = \frac{1 + 0.25z^{-1}}{1 + 0.8z^{-1} - 0.84z^{-2}} = \frac{1 + 0.25z^{-1}}{(1 + 1.4z^{-1})(1 - 0.6z^{-1})} \cdot \frac{z^2}{z^2} = \frac{z(z + 0.25)}{(z + 1.4)(z - 0.6)}$$

- Two zeros at $z = 0$ and $z = -0.25$
- Two poles at $z = -1.4$ and $z = 0.6$



Solution (b)



Find **a stable impulse response** $h[n]$.

- For a stable LTI system, the ROC must include the unit circle. Thus, the ROC is
  - $ROC: 0.6 < |z| < 1.4$
- The impulse response $h[n]$ can be obtained by inverse z-transform of H(z) with this ROC

$$H(z) = \frac{1 + 0.25z^{-1}}{(1 + 1.4z^{-1})(1 - 0.6z^{-1})} = \frac{0.575}{1 - (-1.4)z^{-1}} + \frac{0.425}{1 - 0.6z^{-1}}$$

Left-sided sequence of ROC |z|<1.4

Right-sided sequence of ROC |z|>0.6

$$h[n] = -0.575(-1.4)^n u[-n-1] + 0.425(0.6)^n u[n]$$

Partial Fraction

$$H(z) = \frac{1 + 0.25z^{-1}}{(1 + 1.4z^{-1})(1 - 0.6z^{-1})} = \frac{A}{1 + 1.4z^{-1}} + \frac{B}{1 - 0.6z^{-1}}$$

$$A = (1 + 1.4z^{-1})H(z)\Big|_{z=-1.4} = \frac{1 + 0.25z^{-1}}{1 - 0.6z^{-1}}\Big|_{z=-1.4} = \frac{1 + 0.25\left(\frac{1}{-1.4}\right)}{1 - 0.6\left(\frac{1}{-1.4}\right)} = 0.5750$$

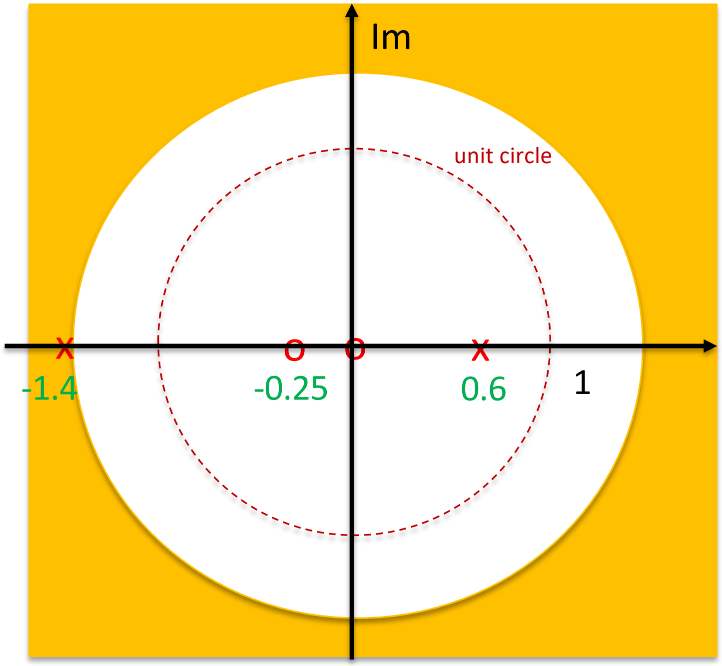
$$B = (1 - 0.6z^{-1})H(z)\Big|_{z=0.6} = \frac{1 + 0.25z^{-1}}{1 + 1.4z^{-1}}\Big|_{z=0.6} = \frac{1 + 0.25\left(\frac{1}{0.6}\right)}{1 + 1.4\left(\frac{1}{0.6}\right)} = 0.425$$

$$H(z) = \frac{1 + 0.25z^{-1}}{(1 + 1.4z^{-1})(1 - 0.6z^{-1})} = \frac{0.575}{1 + 1.4z^{-1}} + \frac{0.425}{1 - 0.6z^{-1}}$$

Solution (c)

Find **a causal impulse response** exist that is both stable and causal?

- For a causal LTI system, the impulse response $h[n]$ is right-sided sequence ROC. Then, the ROC is
  - $ROC: 1.4 < |z|$
- The impulse response $h[n]$ can be obtained by inverse z-transform of H(z) with this ROC



$$H(z) = \frac{0.5750}{1-(-1.4)z^{-1}} + \frac{0.425}{1-0.6z^{-1}} \qquad ROC: 1.4 < |z|$$

$$h[n] = 0.5750(-1.4)^n u[n] + 0.425(0.6)^n u[n]$$

For this transfer function, we cannot achieve both stable and causal as the system is causal, the ROC cannot include the unit circle.