2-D System Theory Based Iterative Learning Control for Linear Continuous Systems With Time Delays

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***Abstract*—This paper presents two-dimensional (2-D) system theory based iterative learning control (ILC) methods for linear continuous multivariable systems with time delays in state or with time delays in input. Necessary and sufficient conditions are given for convergence of the proposed ILC rules. In this paper, we demonstrate that the 2-D linear continuous-discrete Roesser's model can be applied to describe the ILC process of linear continuous time-delay systems. Three numerical examples are used to illustrate the effectiveness of the proposed ILC methods.**

***Index Terms*—Iterative control, linear continuous multivariable system, time delay, two-dimensional (2-D) system theory.**

I. Introduction

ITERATIVE learning control (ILC), firstly introduced in 1984 by Arimoto *et al.* [4], is well known with its ability to determine a control input iteratively so that the tracking of a given reference signal or the output trajectory over a fixed time interval is possible. It has an appealing capability of modifying an unsatisfactory control input signal based on the knowledge of previous operations of the same task. The most common applications of ILC are in the area of robot control in production industries. One of the most attractive features of the ILC is that it requires less *a priori* knowledge about the controlled system in the controller design phase.

ILC has generated considerable research interest over the past years. Until now, there have been a lot of ILC algorithms presented in the area of control systems [4]–[17], [21]–[23]. Among these different methods, the most widely used ILC algorithm is the proportional-integral-derivative (PID)-type approach because it essentially forms a PID-like system. Geng *et al.* [13] pointed out that all PID-type ILC techniques inevitably suffer from a tight restriction, and hence applied the two-dimensional (2-D) system theory to ILC schemes. Although there are certain advantages contributed by the ILC control schemes, the technical difficulty due to the two-dimensionality has always been essential and these problems were addressed in [17]. In fact, one of the main difficulties experienced in the ILC is the establishment of a suitable mathematical model to clearly describe the dynamics of the control system and the behavior of the learning process [13]. It is well known that amid the iterative learning process, the interaction between the system dynamics and the iterative learning process pose an important and challenging issue for ILC research.

In recent years, 2-D system theory was successfully introduced to the ILC approaches [8]–[14], [24], [25]. Contributing by the two independent dynamic processes of the 2-D system, the 2-D model provides an excellent mathematical platform to describe both the dynamics of the control system and the behavior of the learning iteration. In the 2-D system theory based ILC techniques for linear multivariable systems, very promising results have been obtained [8], [9], [11]–[14]. Some of the work focused on the investigation of the ILC techniques for linear discrete multivariable systems [9], [12]–[14]. In [8], the 2-D system theory based ILC techniques were extended to linear continuous multivariable systems. Recently, 2-D system theory was applied to solve the initial ILC condition problem for linear discrete multivariable systems [11]. More importantly, a new type of 2-D system theory based ILC rules for linear continuous systems and linear discrete systems were proposed in [8], [14]. These newly proposed methods are able to drive the control error to zero with only one iteration. All of these 2-D system theory based ILC techniques can be attributed to the rigorous stability theory of linear repetitive process developed in [24], [25].

Despite the promising results on the area of 2-D system theory based ILC, it is worth noting that all these work only focused on linear systems without time delay. Linear time-delay systems, which exhibit more complicated dynamics, have difference in essence from linear delay-free systems. And the study of time-delay systems has become increasingly important over the past years [1]–[3], [19]–[22] because the issue of time delay is often encountered in many practical systems such as actuators, sensors, field networks involved in feedback loops, and the delays introduced by the computation of robotics control. It is worth noting that the existence of time delay quite often degrades the performance of a control system, or even destabilizes the whole system. Hitherto, a lot of achievements have been obtained in the area of time-delay system control, especially in the area of stabilization of time-delay systems [19], [20], and a detailed review can be found in [1]. But most of these existing control approaches for time-delay systems are sensitive to the system uncertainty. They usually require an accurate mathematical model, which is rather difficult or even impossible in most practical applications. Naturally, ILC is found to be a good alternative to deal with the control of time-delay systems, especially when detailed knowledge about the plant is not available. However, until now, there have been only limited works to study this issue [21], [22].

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The main objective of this paper is to further extend the existing 2-D system theory based ILC techniques for linear multivariable systems to linear continuous systems with time delays, including time delays in state and time delays in input. In the study of 2-D system theory based ILC for linear continuous systems with time delays, first we address the ILC systems with only a single time delay. We will then extend the derived results to the cases with multiple time delays. Our strategy is to reconstruct the derived ILC error equations in a compact form of the 2-D linear continuous-discrete Roessor's model so that we can obtain certain convergent ILC rules according to the property of 2-D linear systems.

The organization of this paper is as follows. Section II investigates the 2-D system theory based ILC techniques for linear continuous systems with time delays in state, and Section III dresses the same ILC problem for linear continuous systems with time delays in input. The simulation results are presented in Section IV. Finally, Section V concludes this paper.

II. ILC for Linear Continuous Multivariable Systems With Time-Delayes in State

Consider linear continuous multivariable systems with a single time delay in state performing a given task repeatedly on a finite time interval $[0,T]$. The systems can be described by the following differential equations:

$$\frac{\partial x(t,k)}{\partial t} = Ax(t,k) + A_0 x(t-t_0,k) + Bu(t,k) \quad \text{(1a)}$$
$$y(t,k) = Cx(t,k) \quad \text{(1b)}$$

where k indicates the number of operation cycle, and t is the continuous-time index running from $t=0$ to $t=T$ to complete a cycle. For all $t \in [0,T], x(t,k) \in R^n, u(t,k) \in R^m$ and $y(t,k) \in R^p$ are the state vectors, the input vectors and the output vectors, respectively. t_0 is a time-delay parameter, and A, A_0, B, C are real matrices of appropriate dimensions that are possibly values estimated. The ILC problem that we are dealing with is stated as follows. Given system (1) with initial state $x(t,k), t \in [-t_0, 0]$, and derivable reference output $y_r(t), t \in [0,T]$, iteratively find an appropriate control input $u(t), t \in [0,T]$ such that the system output follows the reference output trajectory for $t \in [0,T]$.

Suppose that a general ILC rule for system (1) is given as

$$u(t,k+1) = u(t,k) + \Delta u(t,k) \quad (2)$$

where Δu denotes modification of the control input. The boundary conditions for the ILC system (1) and (2) are

$$\begin{aligned} x(t,k) &= x_0(t), \qquad t \in [-t_0, 0] \quad \text{for } k = 0,1,2\ldots \\ u(t,0) &= u_0(t), \qquad \text{for } t \in [0,T]. \end{aligned} \quad (3)$$

It is also assumed that $y(0,k) = Cx(0,k) = y_r(0)$. Based on these boundary conditions, our ILC objective is to minimize the tracking error $\|y_r(t) - y(t,k)\|$ for $t = [0,T]$, where $\|\,.\,\|$ represents the matrix norm.

Let us denote

$$e(t,k) = y_r(t) - y(t,k) \quad (4)$$
$$\eta(t,k) = \int_0^t [x(\tau,k+1) - x(\tau,k)]\, d\tau. \quad (5)$$

Considering that $x(t,k+1) - x(t,k) = 0$ and $e(0,k) = 0$ for $t \in [-t_0, 0]$ and $k = 0,1,2,\ldots$, we have

$$\begin{aligned} \frac{\partial \eta(t,k)}{\partial t} &= x(t,k+1) - x(t,k) \\ &= \int_0^t \frac{\partial x(\tau,k+1)}{\partial \tau}\, d\tau - \int_0^t \frac{\partial x(\tau,k)}{\partial \tau}\, d\tau \\ &= A\int_0^t [x(\tau,k+1) - x(\tau,k)]\, d\tau \\ &\quad + A_0 \int_0^t [x(\tau - t_0, k+1) - x(\tau - t_0, k)]\, d\tau \\ &\quad + B\int_0^t \Delta u(\tau,k)\, d\tau \\ &= A\eta(t,k) + A_0\eta(t-t_0,k) + B\int_0^t \Delta u(\tau,k)\, d\tau. \end{aligned} \quad (6)$$

And from (4) and (6)

$$\begin{aligned} e(t,k+1) - e(t,k) &= -[y(t,k+1) - y(t,k)] \\ &= -C[x(t,k+1) - x(t,k)] \\ &= -CA\eta(t,k) - CA_0\eta(t-t_0,k) \\ &\quad - CB\int_0^t \Delta u(\tau,k)\, d\tau. \end{aligned} \quad (7)$$

Let

$$\Delta u(t,k) = K_1 \frac{\partial \eta(t,k)}{\partial t} + K_2 \frac{\partial \eta(t-t_0,k)}{\partial t} + K_3 \frac{\partial e(t,k)}{\partial t} \quad (8)$$

then, (6) and (7) become

$$\begin{aligned} \frac{\partial \eta(t,k)}{\partial t} &= (A + BK_1)\eta(t,k) + (A_0 \\ &\quad + BK_2)\eta(t-t_0,k) + BK_3 e(t,k) \end{aligned} \quad (9)$$

$$\begin{aligned} e(t,k+1) &= -(CA + CBK_1)\eta(t,k) \\ &\quad - (CA_0 + CBK_2)\eta(t-t_0,k) \\ &\quad + (I - CBK_3)e(t,k) \end{aligned} \quad (10)$$

where (and afterwards) I is simply used to represent an identity matrix of appropriate order. Furthermore, let us make the following matrix denotation:

$$\tilde{\eta}(t,k) = \begin{bmatrix} \eta(t,k) \\ \eta(t-t_0,k) \\ \eta(t-2t_0,k) \\ \cdots \\ \eta\left(t - \mathrm{int}\left(\frac{t}{t_0}\right)\cdot t_0, k\right) \end{bmatrix}$$

$$\tilde{e}(t,k) = \begin{bmatrix} e(t,k) \\ e(t-t_0,k) \\ e(t-2t_0,k) \\ \cdots \\ e\left(t - \mathrm{int}\left(\frac{t}{t_0}\right)\cdot t_0, k\right) \end{bmatrix}$$

where (and afterwards) $\mathrm{int}(\cdot)$ represents the integer part of value

$$\tilde{A}=\begin{bmatrix} A+BK_1 & A_0+BK_2 & 0 & \cdots & 0 \\ 0 & A+BK_1 & A_0+BK_2 & \cdots & 0 \\ 0 & 0 & A+BK_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A+BK_1 \end{bmatrix}$$

$$\tilde{B}=\begin{bmatrix} BK_3 & 0 & 0 & \cdots & 0 \\ 0 & BK_3 & 0 & \cdots & 0 \\ 0 & 0 & BK_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & BK_3 \end{bmatrix}$$

$$\tilde{C}=\begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \cdots & 0 \\ 0 & 0 & C & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C \end{bmatrix}$$

$$\tilde{\omega}(t,k)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ (A_0+BK_2)\eta\left(t-\mathrm{int}\left(\frac{t}{t_0}+1\right)\cdot t_0,k\right) \end{bmatrix}.$$

Then, according to (9) and (10), the following equation in the compact form can be derived:

$$\begin{bmatrix} \frac{\partial\tilde{\eta}(t,k)}{\partial t} \\ \tilde{e}(t,k+1) \end{bmatrix}=\begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{C}\tilde{A} & I-\tilde{C}\tilde{B} \end{bmatrix}\begin{bmatrix} \tilde{\eta}(t,k) \\ \tilde{e}(t,k) \end{bmatrix}+\begin{bmatrix} \tilde{\omega}(t,k) \\ -\tilde{C}\tilde{\omega}(t,k) \end{bmatrix}. \quad (11)$$

From the denotation of $\tilde{\omega}(t,k)$ and the boundary conditions (3), it can be concluded that $\tilde{\omega}(t,k)=0$. Consequently, the 2-D linear continuous-discrete system (11) is changed into a simpler form of Roeesor's type model [18]

$$\begin{bmatrix} \frac{\partial\tilde{\eta}(t,k)}{\partial t} \\ \tilde{e}(t,k+1) \end{bmatrix}=\begin{bmatrix} \tilde{A} & \tilde{B} \\ -\tilde{C}\tilde{A} & I-\tilde{C}\tilde{B} \end{bmatrix}\begin{bmatrix} \tilde{\eta}(t,k) \\ \tilde{e}(t,k) \end{bmatrix}. \quad (12)$$

The boundary conditions for the 2-D system (12) are $\tilde{\eta}(0,k)=0$ for $k=0,1,2,\ldots$ and finite $\tilde{e}(t,0)$ for $t\in[0,T]$.

Lemma 1: For a 2-D linear continuous-discrete system

$$\begin{bmatrix} \frac{\partial x(t,k)}{\partial t} \\ y(t,k+1) \end{bmatrix}=\begin{bmatrix} A_1 & A_2 \\ A_3 & A_4 \end{bmatrix}\begin{bmatrix} x(t,k) \\ y(t,k) \end{bmatrix} \quad (13)$$

where $x(t,k)\in R^{n_1}, y(t,k)\in R^{n_2}, A_1\in R^{n_1\times n_1}, A_2\in R^{n_1\times n_2}, A_3\in R^{n_2\times n_1}$, and $A_4\in R^{n_2\times n_2}$, and $x(0,k)=0$ are the boundary conditions for $k=0,1,2,\ldots$ and finite $y(t,0)$ for $t>0$. Then, for all t which belong to the interval $[0,T], [\begin{smallmatrix} x(t,k) \\ y(t,k) \end{smallmatrix}]\to 0$ as $k\to\infty$ iff the matrix A_4 is stable.

The proof can be referred to ([8], Proof of Theorem 1).

It can be noticed that $\lim_{k\to\infty}[\begin{smallmatrix} \tilde{\eta}(t,k) \\ \tilde{e}(t,k) \end{smallmatrix}]=0$ for $t\in[0,T]$ is equivalent to $\lim_{k\to\infty}[\begin{smallmatrix} \eta(t,k) \\ e(t,k) \end{smallmatrix}]=0$ for $t\in[0,T]$. According to Lemma 1 and the denotation of $\tilde{B},\tilde{C}$ in the system (12), $\lim_{k\to\infty}[\begin{smallmatrix} \eta(t,k) \\ e(t,k) \end{smallmatrix}]=0$ for $t\in[0,T]$iff there exists a matrix K_3 to stabilize the matrix $I-CBK_3$. And it is easy to show that a matrix K_3 exists that stabilize the matrix $I-CBK_3$ iff matrix CB has full-row rank. On the other hand, $\lim_{k\to\infty}[\begin{smallmatrix} \eta(t,k) \\ e(t,k) \end{smallmatrix}]=0$ for $t\in[0,T]$ have nothing to do with the matrices K_1 and K_2. Therefore, for simplicity, we let $K_1=K_2=0$ and $K=K_3$ in (8), and formulate the following theorem.

Theorem 1: For a linear continuous multivariable system (1) with time delay in state, suppose that the desired output $y_r(t)$ is derivable. There exists an ILC rule

$$u(t,k+1)=u(t,k)+K\frac{\partial e(t,k)}{\partial t} \quad (14)$$

to make $\lim_{k\to\infty}[\begin{smallmatrix} \eta(t,k) \\ e(t,k) \end{smallmatrix}]=0$ for $t\in[0,T]$ iff the matrix CB has full-row rank.

In addition, according to Lemma 1, the matrixes K_1, K_2, and K_3 of (8) can also be selected as other forms in order to obtain suitable ILC rules with some special properties. The only requirement to K_1, K_2, and K_3 is to ensure the stability of matrix $I-CBK_3$. When the system matrices A, A_0, B, C are accurately known, let

$$\begin{aligned} K_1 &= -(CB)^T[CB(CB)^T]^{-1}CA \\ K_2 &= -(CB)^T[CB(CB)^T]^{-1}CA_0 \\ K_3 &= (CB)^T[CB(CB)^T]^{-1}. \end{aligned}$$

It can be shown from (10) that we always have $e(t,1)=0$ for $t\in[0,T]$ no matter what $e(t,0)$ is. Thus, the following theorem has been proved.

Theorem 2: For a linear continuous multivariable system (1) with time delay in state, suppose that the desired output $y_r(t)$ is derivable. An ILC rule

$$\begin{aligned} u(t,k+1) &= u(t,k)+K_1\cdot[x(t,k+1)-x(t,k)]+K_2 \\ &\quad \cdot[x(t-t_0,k+1)-x(t-t_0,k)]+K_3\cdot\frac{\partial e(t,k)}{\partial t} \end{aligned} \quad (15)$$

where

$$\begin{aligned} K_1 &= -(CB)^T[CB(CB)^T]^{-1}CA \\ K_2 &= -(CB)^T[CB(CB)^T]^{-1}CA_0 \\ K_3 &= (CB)^T[CB(CB)^T]^{-1} \end{aligned}$$

drives the control error to zero for the desired output at the interval $t\in[0,T]$ after only one learning iteration iff the matrix CB has full-row rank.

In Fig. 1, the proposed ILC rule (15) is briefly illustrated. Undoubtedly, Theorem 2 shows that the ILC rule (15) exhibits the fastest iterative convergent rate. It is noted that $x(t,k+1)$ is not available in (15), estimation or computation of $x(t,k+1)$ should be made. In practice, $x(t,k+1)-x(t,k)$ may be replaced by $x(t,k)-x(t,k-1)$.

In many practical applications of Theorem 2, the actual values on parameters A, A_0, B, C are not available and we have to rely on their estimates. Despite of this situation, K_3, that is evaluated from the estimates of B, C, is able to make the eigenvalues of $I-CBK_3$ inside the unite circle. Using the result shown in (12) and Lemma 1,

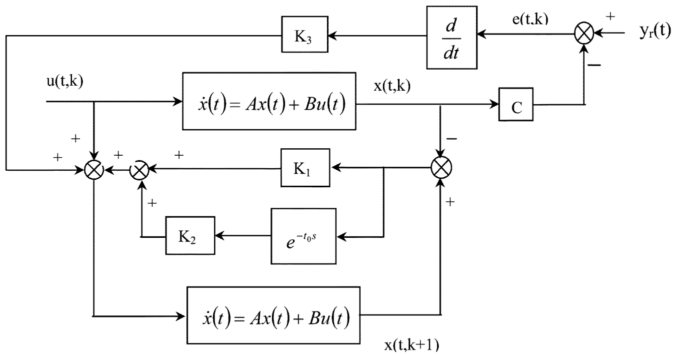

Fig. 1. Block diagram of the proposed ILC rule (15).

it is clear that the ILC rule (15) is still convergent. The robustness of the ILC rule (15) is illustrated by using the following example 1. In the following, Theorem 3 proves the ILC rule (14) is also applicable to the ILC problem for linear continuous multivariable systems with multiple time delays in state.

Theorem 3: For the following linear continuous multivariable system with multiple time delays in state

$$\frac{\partial x(t,k)}{\partial t} = Ax(t,k) + \sum_{s=0}^{l} A_s x(t-t_s,k) + Bu(t,k) \quad (16a)$$

$$y(t,k) = Cx(t,k). \quad (16b)$$

Suppose that the desired output $y_r(t)$ is derivable. There exists an ILC rule (14) to make $\lim_{k\to\infty}[\begin{smallmatrix}\eta(t,k)\\ e(t,k)\end{smallmatrix}] = 0$ for $t \in [0,T]$ iff the matrix CB has full-row rank.

Proof: Suppose that the initial ILC conditions for system (16) are $x(t,k) = x_0(t), t \in \lfloor -\max_{0\le s\le l}\{t_s\}, 0\rfloor$, and $y(0,k) = y_r(0)$ for $k = 0,1,2\ldots$. Similar to the 2-D representation of (6) and (7) for linear time-delay system (1), the ILC error model of linear multiple time-delay system (16) using the ILC rule (14) can be expressed as

$$\frac{\partial \eta(t,k)}{\partial t} = A\eta(t,k) + \sum_{s=0}^{l} A_s \eta(t-t_s,k) + BKe(t,k) \quad (17)$$

$$e(t,k+1) = -CA\eta(t,k) - C\sum_{s=0}^{l} A_s\eta(t-t_s,k) + (I-CBK)e(t,k). \quad (18)$$

Let $c_s (0 \le s \le l)$ be nonnegative integers. Then, for each fixed $t \in [0,T]$, the number of $t - \sum_{s=0}^{l} c_s t_s$, which satisfy $t - \sum_{s=0}^{l} c_s t_s \ge 0$, is finite. Based on the value of $t - \sum_{s=0}^{l} c_s t_s$ from large to small, we list a series of $\eta(t - \sum_{s=0}^{l} c_s t_s, k)$ as a columned-vector $\tilde{\eta}(t,k)$, and a series of $e(t - \sum_{s=0}^{l} c_s t_s, k)$ as a columned-vector $\tilde{e}(t,k)$, where $t - \sum_{s=0}^{l} c_s t_s \ge 0$.

Considering that $\eta(t - \sum_{s=0}^{l} c_s t_s, k) = 0$ as $t - \sum_{s=0}^{l} c_s t_s < 0$, from (17) and (18), we have

$$\begin{bmatrix} \frac{\partial \tilde{\eta}(t,k)}{\partial t} \\ \tilde{e}(t,k+1) \end{bmatrix} = \begin{bmatrix} \Phi_1 & \Phi_2 \\ \Phi_3 & \Phi_4 \end{bmatrix} \begin{bmatrix} \tilde{\eta}(t,k) \\ \tilde{e}(t,k) \end{bmatrix} \quad (19)$$

where Φ_1 and Φ_3 are upper triangular matrices; Φ_2 and Φ_4 are diagonal matrices, and all the diagonal elements of Φ_4 are $I - CBK$. According to Lemma 1, Theorem 3 is proved.

Remark: Regarding the ILC problem for the following linear continuous systems with multiple time delays in state and input

$$\frac{\partial x(t,k)}{\partial t} = Ax(t,k) + \sum_{s=0}^{l} A_s x(t-t_s,k) + Bu(t-\tau,k) \quad (20a)$$

$$y(t,k) = Cx(t,k) \quad (20b)$$

if we let $u'(t,k) = u(t-\tau,k)$, then, the system (20) is changed to the form of the linear continuous multivariable systems (16) with multiple time delays in state. As a result, our proposed ILC rule (14) can be easily modified for the application to the linear continuous systems (20) with multiple time delays in state and input.

III. ILC for Linear Continuous Multivariable Systems With Time-Delays in Input

Next, we consider the ILC problem for the linear continuous multivariable system with a single time delay in input

$$\frac{\partial x(t,k)}{\partial t} = Ax(t,k) + B_0 u(t-t_0,k) + Bu(t,k) \quad (21a)$$

$$y(t,k) = Cx(t,k). \quad (21b)$$

Our objective is to find an appropriate control input $u(t), t \in [-t_0, T]$ iteratively such that the system output follows the derivable reference output $y_r(t)$ at the time interval $t \in [0,T]$. The boundary conditions for the 2-D linear continuous-discrete system (21) with the general ILC rule (2) are

$$x(0,k) = x_0(0), \qquad y(0,k) = y_r(0) \quad \text{for } k = 0,1,2\ldots$$
$$u(t,0) = u_0(t), \qquad \text{for } t \in [-t_0, T]. \quad (22)$$

Using the definition (4), (5), and (21), and following a similar deriving procedure of (6), (7), we have

$$\frac{\partial \eta(t,k)}{\partial t} = A\eta(t,k) + B_0 \int_0^t \Delta u(\tau - t_0, k)\, d\tau + B\int_0^t \Delta u(\tau,k)\, d\tau \quad (23)$$

$$e(t,k+1) - e(t,k) = -CA\eta(t,k) - CB_0 \int_0^t \Delta u(\tau - t_0,k)\, d\tau - CB\int_0^t \Delta u(\tau,k)\, d\tau. \quad (24)$$

Considering (21) and its boundary condition (22), we are able to denote that $\eta(t,k) = 0$ and $e(t,k) = 0$ at the time interval $t \in [-t_0, 0]$ for $k = 0,1,2\ldots$.

Let

$$\Delta u(t,k) = K_1 \frac{\partial \eta(t,k)}{\partial t} + K_2 \frac{\partial e(t,k)}{\partial t} \quad (25)$$

then, (23) and (24) become

$$\frac{\partial \eta(t,k)}{\partial t} = (A+BK_1)\eta(t,k) + BK_2 e(t,k) + B_0K_1\eta(t-t_0,k) + B_0K_2 e(t-t_0,k) \quad (26)$$

$$e(t,k+1) = -C(A+BK_1)\eta(t,k) + (I-CBK_2)e(t,k) - CB_0K_1\eta(t-t_0,k) - CB_0K_2e(t-t_0,k). \quad (27)$$

Let us make the following matrix denotation:

$$\bar{\eta}(t,k) = \tilde{\eta}(t,k)$$
$$\bar{e}(t,k) = \tilde{e}(t,k)$$

$$\bar{A} = \begin{bmatrix} A+BK_1 & B_0K_1 & 0 & \cdots & 0 \\ 0 & A+BK_1 & B_0K_1 & \cdots & 0 \\ 0 & 0 & A+BK_1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & A+BK_1 \end{bmatrix}$$

$$\bar{B} = \begin{bmatrix} BK_2 & B_0K_2 & 0 & \cdots & 0 \\ 0 & BK_2 & B_0K_2 & \cdots & 0 \\ 0 & 0 & BK_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & BK_2 \end{bmatrix}$$

$$\bar{C} = \begin{bmatrix} C & 0 & 0 & \cdots & 0 \\ 0 & C & 0 & \cdots & 0 \\ 0 & 0 & C & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & C \end{bmatrix}$$

and the equation shown at the bottom of the page.

Then, according to (26) and (27), the following equation in the compact form can be derived:

$$\begin{bmatrix} \frac{\partial \bar{\eta}(t,k)}{\partial t} \\ \bar{e}(t,k+1) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ -\bar{C}\bar{A} & I-\bar{C}\bar{B} \end{bmatrix} \begin{bmatrix} \bar{\eta}(t,k) \\ \bar{e}(t,k) \end{bmatrix} + \begin{bmatrix} \bar{\omega}(t,k) \\ -\bar{C}\bar{\omega}(t,k) \end{bmatrix}. \quad (28)$$

From the expression of $\bar{\omega}(t,k)$, we know $\bar{\omega}(t,k) = 0$. Consequently, the 2-D linear continuous-discrete system (28) is changed into a simpler form of Roessor's type model

$$\begin{bmatrix} \frac{\partial \bar{\eta}(t,k)}{\partial t} \\ \bar{e}(t,k+1) \end{bmatrix} = \begin{bmatrix} \bar{A} & \bar{B} \\ -\bar{C}\bar{A} & I-\bar{C}\bar{B} \end{bmatrix} \begin{bmatrix} \bar{\eta}(t,k) \\ \bar{e}(t,k) \end{bmatrix}. \quad (29)$$

The boundary conditions for the 2-D system (29) are $\bar{\eta}(0,k) = 0$ for $k = 0,1,2,\ldots$ and finite $\bar{e}(t,0)$ for $t \in [0,T]$. Similar to the case of ILC for linear continuous system (1) with time delay in state, it is also noticed that $\lim_{k\to\infty}[\begin{smallmatrix}\bar{\eta}(t,k)\\ \bar{e}(t,k)\end{smallmatrix}] = 0$ for $t \in [0,T]$ is equivalent to $\lim_{k\to\infty}[\begin{smallmatrix}\eta(t,k)\\ e(t,k)\end{smallmatrix}] = 0$ for $t \in [0,T]$. According to Lemma 1 and the denotation of $\bar{B}, \bar{C}$ in the system (29), $\lim_{k\to\infty}[\begin{smallmatrix}\eta(t,k)\\ e(t,k)\end{smallmatrix}] = 0$ for $t \in [0,T]$iff there exists a matrix K_2 to stabilize the matrix $I - CBK_2$. And, $\lim_{k\to\infty}[\begin{smallmatrix}\eta(t,k)\\ e(t,k)\end{smallmatrix}] = 0$ for $t \in [0,T]$ has nothing to do with the matrix K_1. Letting $K_1 = 0$ and $K = K_2$ in (25), we can then formulate the following theorem.

TABLE I
(EXAMPLE 2) TOTAL SQUARED ERROR OF ILC TRACKING PERFORMANCE AT FOUR DIFFERENT TIME INTERVALS $t \in [0,1]$, AND AT DIFFERENT ITERATIONS USING ILC RULE (31)

	Time Interval [0, 0.25]	Time Interval [0.25, 0.5]	Time Interval [0.5, 0.75]	Time Interval [0.75, 1]
Iteration 0	$[0.016 \quad 0.034]^T$	$[0.278 \quad 0.162]^T$	$[0.698 \quad 0.160]^T$	$[0.315 \quad 0.028]^T$
Iteration 1	$[0 \quad 0]^T$	$[0.046 \quad 0.014]^T$	$[0.213 \quad 0.203]^T$	$[0.192 \quad 0.462]^T$
Iteration 2	$[0 \quad 0]^T$	$[0 \quad 0]^T$	$[0.016 \quad 0.024]^T$	$[0.316 \quad 0.180]^T$
Iteration 3	$[0 \quad 0]^T$	$[0 \quad 0]^T$	$[0 \quad 0]^T$	$[0.033 \quad 0.013]^T$
Iteration ≥4	$[0 \quad 0]^T$	$[0 \quad 0]^T$	$[0 \quad 0]^T$	$[0 \quad 0]^T$

Theorem 4: For a linear continuous multivariable system (21) with time delay in input, suppose that the desired output $y_r(t)$ is derivable. There exists an ILC rule

$$u(t,k+1) = u(t,k) + K\frac{\partial e(t,k)}{\partial t} \quad (30)$$

to make $\lim_{k\to\infty}[\begin{smallmatrix}\eta(t,k)\\ e(t,k)\end{smallmatrix}] = 0$ for $t \in [0,T]$ iff the matrix CB has full-row rank.

In addition, under the stability requirement of matrix $I - CBK_2$, the matrices K_1 and K_2 in (25) can also be selected as other forms in order to obtain suitable ILC rules with some special properties. It is worth noting that unlike the case of Theorem 2 for linear continuous system (1) with time delay in state, there normally does not exist an ILC rule in the form of (25) to make $e(t,1) = 0$ for $t \in [0,T]$ because of the effect of time delay in the input. But it is important to point out that (27) demonstrates a rapid and interesting convergence characteristic that when $K_1 = -(CB)^T[CB(CB)^T]^{-1}CA$, and $K_2 = (CB)^T[CB(CB)^T]^{-1}$, the control error of the desired output is able to converge to zero in an ordered way of a time interval, which is the unit delay time, when the number of learning iteration increases. This effect can be best demonstrated in later results of Table I. This convergence characteristic can be expressed in the following theorem, and the corresponding ILC rule (31) is illustrated in Fig. 2.

Theorem 5: For a linear continuous multivariable system (21) with time delay in input, suppose that the matrix C has full-column rank, and the desired output $y_r(t)$ is derivable. There exists an ILC rule

$$u(t,k+1) = u(t,k) + K_1\cdot[x(t,k+1)-x(t,k)] + K_2\cdot\frac{\partial e(t,k)}{\partial t} \quad (31)$$

$$\bar{\omega}(t,k) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \cdots \\ B_0K_1\eta\left(t - \mathrm{int}\left(\frac{t}{t_0}+1\right)\cdot t_0, k\right) + B_0K_2 e\left(t - \mathrm{int}\left(\frac{t}{t_0}+1\right)\cdot t_0, k\right) \end{bmatrix}$$

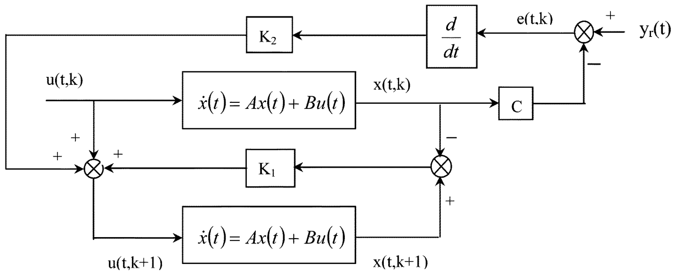

Fig. 2. Block diagram of ILC rule (31).

where

$$K_1 = -(CB)^T[CB(CB)^T]^{-1}CA$$
$$K_2 = (CB)^T[CB(CB)^T]^{-1}$$

to make $e(t,l)=0$ at the interval $t\in[0,(k+1)t_0]$ for $l\geq k+1$ iff the matrix CB has full-row rank.

Proof (Mathematical induction): We have $\eta(t-t_0,l-1)=e(t-t_0,l-1)=0$ at the interval $t\in[0,t_0]$ for $l\geq 1$ according to the initial ILC conditions. From the denotation of K_1, K_2 and error (27), it can be derived that $e(t,l)=0$ at the interval $t\in[0,t_0]$ for $l\geq 1$ iff the matrix CB has full-row rank. So the statement is true for $k=0$.

Assume that as $k=n$, we have $e(t,l)=0$ at the interval $t\in[0,(n+1)t_0]$ for $l\geq n+1$. Then let $k=n+1$, we prove $e(t,l)=0$ at the interval $t\in[0,(n+2)t_0]$ for $l\geq n+2$. As $t\in[0,(n+2)t_0]$ and $l\geq n+2$, it has been assumed that $e(t-t_0,l-1)=0$ and $e(t-t_0,l)=0$, namely, $y(t-t_0,l-1)=y(t-t_0,l)$. If the matrix C has full-column rank, we obtain $x(t-t_0,l-1)=x(t-t_0,l)$. Furthermore, it is derived that $\eta(t-t_0,l-1)=0$. From the denotation of K_1,K_2 and (27), we have $e(t,l)=0$ at the interval $t\in[0,(n+2)t_0]$ for $l\geq n+2$ iff the matrix CB has full-row rank. Theorem 5 is proved.

Theorem 5 ensures that the ILC rule (31) can drive the control error to zero at the interval $t\in[0,T]$ after a finite number of iteration. Similar to Theorem 2, a strategy of computing $x(t,k+1)$ in ILC rule (31) should be employed in practice.

Additionally, from the ILC rule (25) and the extended denotation that $x(t,k+1)=x(t,k)$ and $e(t,k)=0$ at the time interval $t\in[-t_0,0]$ for $k=0,1,2\cdots$, we have $u(t,k+1)=u(t,k)=u_0(t), t\in[-t_0,0]$. Initial input $u_0(t)$ can be randomly selected, thus, there exist much kinds of control input that can drive the system output to the desired reference output. That is, a distinguished character that differ the control system with time delay in the input from other kinds of control system.

Similar to the case of Theorem 2, Theorem 5 also requires accurate information on parameters A,B,C. In many practical applications, only estimates of A,B,C are given. Despite of this situation, K_2, that is elaborated from the estimates, is able to make the eigenvalues of $I-CBK_2$ inside the unite circle. The ILC rule (31) is still convergent but with a slightly lower convergent rate.

Theorem 4 can also be extended to more general linear continuous systems that include multiple time delays in input. As a result, the following theorem is obtained.

Theorem 6: For the following linear continuous multivariable system with multiple time delays in input

$$\frac{\partial x(t,k)}{\partial t} = Ax(t,k) + \sum_{s=0}^{l} B_s u(t-t_s,k) \tag{32a}$$
$$y(t,k) = Cx(t,k) \tag{32b}$$

where $t_0 > t_1 > \cdots > t_l \geq 0$. Suppose that the desired output $y_r(t)$ is derivable. There exists an ILC rule

$$u(t,k+1) = u(t,k) + K\frac{\partial e(t+t_l,k)}{\partial t}, \qquad t\in[-t_0, T-t_l] \tag{33}$$

to make $\lim_{k\to\infty}\begin{bmatrix}\eta(t,k)\\ e(t,k)\end{bmatrix}=0$ for $t\in[0,T]$ iff the matrix CB_l has full-row rank.

Proof: Let us extend the denotation of the initial ILC conditions for system (32) to $x(t,k)=x_0(t)$ and $e(t,k)=0$ at the time interval $t\in[-t_0+t_l,0]$ for $k=0,1,2,\ldots$. Similar to the 2-D representation of (23) and (24) for linear time-delay system (21), the ILC error model of linear system (32) with multiple time delays can be derived as

$$\frac{\partial \eta(t,k)}{\partial t} = A\eta(t,k) + \sum_{s=0}^{l} B_s \int_0^t \Delta u(\tau - t_s, k)\, d\tau \tag{34}$$

$$e(t,k+1)-e(t,k) = -CA\eta(t,k) - C\sum_{s=0}^{l} B_s\int_0^t \Delta u(\tau-t_s,k)\,d\tau. \tag{35}$$

Applying $u(t,k+1)=u(t,k)+K(\partial e(t+t_l,k)/\partial t)$ to (34) and (35), we have

$$\frac{\partial \eta(t,k)}{\partial t} = A\eta(t,k) + \sum_{s=0}^{l-1} B_sKe(t-t_s+t_l,k) + B_lKe(t,k) \tag{36}$$

$$e(t,k+1) = -CA\eta(t,k) - C\sum_{s=0}^{l-1}B_sKe(t-t_s+t_l,k) + (I-CB_lK)e(t,k). \tag{37}$$

Let $c_s (0\leq s\leq l)$ be nonnegative integers, and satisfy $\sum_{s=0}^{l-1} c_s = c_l$. Then, for each fixed $t\in[0,T]$, the number of $t-\sum_{s=0}^{l-1}c_st_s + c_lt_l, (t-\sum_{s=0}^{l-1}c_st_s + c_lt_l \geq 0)$, is finite due to $t_l = \min_{0\leq s\leq l}\{t_s\}$. Based on the value of $t-\sum_{s=0}^{l-1}c_st_s+c_lt_l\geq 0$ from large to small, we list a series of $\eta(t-\sum_{s=0}^{l-1}c_st_s+c_lt_l,k)$ as a columned-vector $\bar{\eta}(t,k)$, and a series of $e(t-\sum_{s=0}^{l-1}c_st_s+c_lt_l,k)$ as a columned-vector $\bar{e}(t,k)$, where $t-\sum_{s=0}^{l-1}c_st_s+c_lt_l\geq 0$. Considering that $e(t-\sum_{s=0}^{l-1}c_st_s+c_lt_l,k)=0$ as $t-\sum_{s=0}^{l-1}c_st_s+c_lt_l<0$, from (36) and (37), we have

$$\begin{bmatrix}\frac{\partial\bar{\eta}(t,k)}{\partial t}\\ \bar{e}(t,k+1)\end{bmatrix} = \begin{bmatrix}\Psi_1 & \Psi_2\\ \Psi_3 & \Psi_4\end{bmatrix}\begin{bmatrix}\bar{\eta}(t,k)\\ \bar{e}(t,k)\end{bmatrix}$$

where Ψ_1 and Ψ_3 are diagonal matrices; Ψ_2 and Ψ_4 are upper triangular matrices, and all the diagonal elements of Ψ_4 are $I-CB_lK$. According to Lemma 1, Theorem 6 is proved.

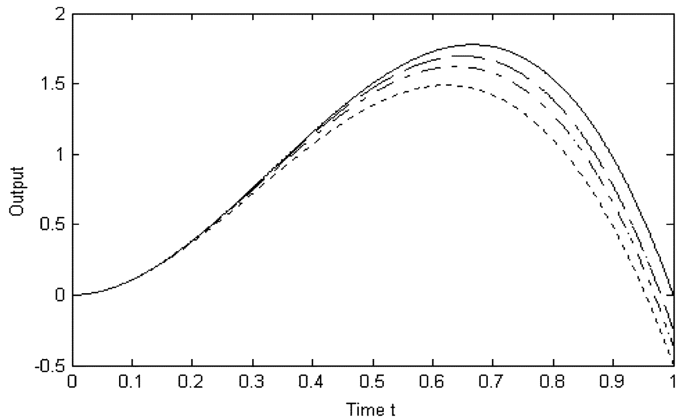

Fig. 3. (Example 1) Outputs of the ILC system for the linear time-delay system (38) using ILC rule (14). The dotted, dashed-dotted, and dashed lines represent the ILC system outputs $y_1(t)$ when the ILC rule has executed 3, 4, and 5 times, respectively, and the solid line represents the desired output $y_{r1}(t)$.

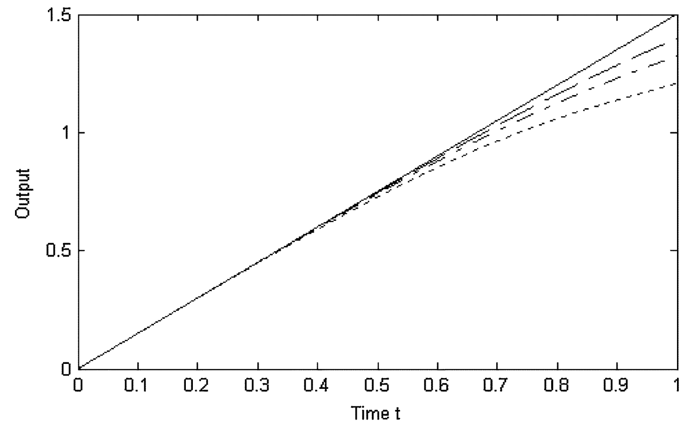

Fig. 4. (Example 1) Outputs of the ILC system for the linear time-delay system (38) using ILC rule (14). The dotted, dashed-dotted, and dashed lines represent the ILC system outputs $y_2(t)$ when the ILC rule has executed 6, 7, 8 times, respectively, and the solid line represents the desired output $y_{r2}(t)$.

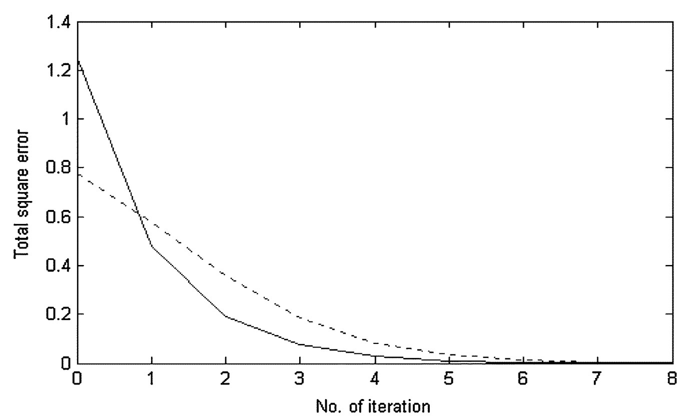

Fig. 5. (Example 1) It shows the total squared error at the interval $t \in [0, 1]$ at different numbers of iterations using ILC rule (14). The solid line represents the curve of total square error of ILC system output $y_1(t)$ to $y_{r1}(t)$, and the dotted line represents the curve of total squared error of the ILC system output $y_2(t)$ to $y_{r2}(t)$.

IV. Simulation Studies

Example 1: Consider an ILC problem of the following linear continuous multivariable system with time delay in state

$$\dot{x}(t) = \begin{bmatrix} -3 & 1 & 0 \\ -2 & -1 & 2 \\ 0 & 1 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -0.5 \\ 0.5 & 0 & 0.5 \end{bmatrix} x(t-0.5) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -1 & 1 \end{bmatrix} u(t) \tag{38a}$$

$$y(t) = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} x(t) \tag{38b}$$

where $x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^T$ and $y(t) = [y_1(t)\ y_2(t)]^T$. The initial value of state variable is $x(t) = [t\ t\ t]^T$ for $t \in [-0.5, 0]$. The desired output $y_r(t)$ is described as

$$y_r(t) = \begin{bmatrix} y_{r1}(t) \\ y_{r2}(t) \end{bmatrix} = \begin{bmatrix} 12t^2(1-t) \\ 1.5t \end{bmatrix}, \qquad t \in [0, 1].$$

In the ILC process of system (38), $x(t, k) = [t\ t\ t]^T$ for $t \in [-0.5, 0]$ and $k = 0, 1, 2, \ldots$, and $u(t, 0) = [0\ 0]^T$ for $t \in [0, 1]$ are assumed. The accuracy of ILC is evaluated by the following total square error of tracking

$$\mathrm{EE} = \int_0^1 [y_r(\tau) - y(\tau)]^2\, d\tau.$$

First, we apply the proposed ILC rule (14) to a linear continuous system (38) with time delay in state, and set $K = \left[\begin{smallmatrix} 0.8 & 0 \\ 0 & 0.8 \end{smallmatrix}\right]$, which can stabilize the matrix $I - CBK$. In this example, two different types of outputs were studied. Figs. 3 and 4 show the tracking performance of the ILC system for the two different outputs at the interval $t \in [0, 1]$ when the ILC rule (14) is iteratively executed for different times. Also, Fig. 5 shows the curves of the total squared error of tracking in the process of ILC rule (14) being iteratively executed. From Figs. 3–5, it can be noticed that the convergence rate is high and the outputs are capable of approaching the desired trajectories accurately within few iterations.

Next, the ILC rule (15) with

$$K_1 = -(CB)^T [CB(CB)^T]^{-1} CA$$
$$K_2 = -(CB)^T [CB(CB)^T]^{-1} CA_0$$
$$K_3 = (CB)^T [CB(CB)^T]^{-1}$$

is used. As expected, it practically drives the tracking error to zero for the desired output at the interval $t \in [0, 1]$ after only one learning iteration. Finally, provided that the accurate information on parameters A, A_0, B, C in system (38) is unavailable, and only estimation is given as

$$\hat{A} = \begin{bmatrix} -2.8 & 1 & 0.1 \\ -2 & -1.2 & 1.9 \\ 0 & 1.2 & -2.1 \end{bmatrix}$$

$$\hat{A}_0 = \begin{bmatrix} 1.08 & -1 & 0.1 \\ 0.12 & 1 & -0.55 \\ 0.4 & 0.1 & 0.5 \end{bmatrix}$$

$$\hat{B} = \begin{bmatrix} 0.08 & 0.9 \\ -1.1 & 0 \\ -1.2 & 1 \end{bmatrix}$$

$$\hat{C} = \begin{bmatrix} 1 & 0.02 & -0.91 \\ 1 & 0.12 & -0.05 \end{bmatrix}$$

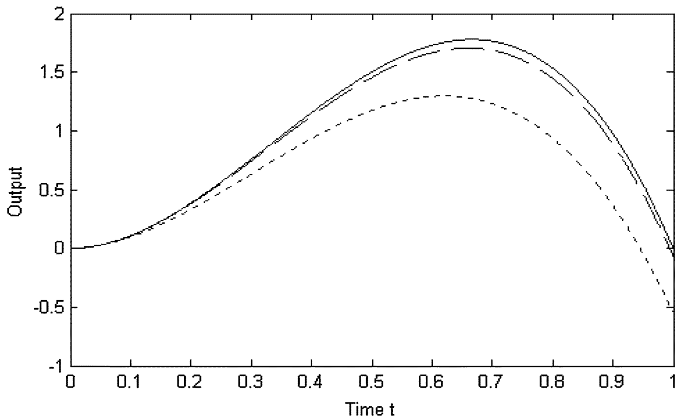

Fig. 6. (Example 1) Outputs of the ILC system for the linear time-delay system (38) using ILC rule (15) with estimated system parameters. The dotted, and dashed lines represent the ILC system outputs $y_1(t)$ as the ILC rule (15) is iteratively executed 1, 2 times, respectively. The solid line represents the desired output $y_{r1}(t)$.

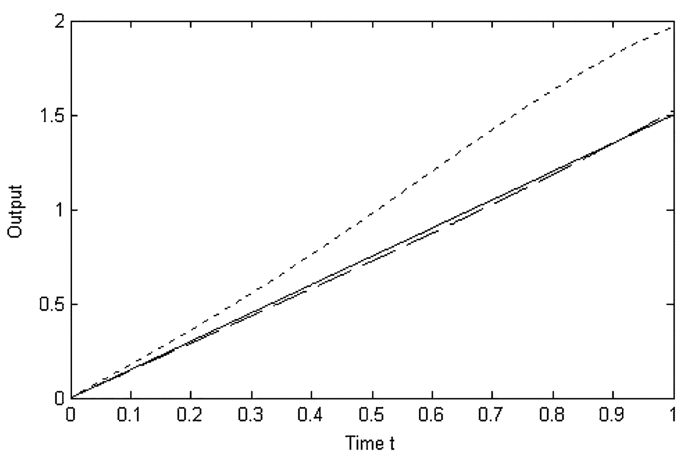

Fig. 7. (Example 1) Outputs of the ILC system for the linear time-delay system (38) using ILC rule (15) with estimated system parameters. The dotted, and dashed lines represent the ILC system outputs $y_2(t)$ as the ILC rule (15) is iteratively executed 1, 2 times, respectively. The solid line represents the desired output $y_{r2}(t)$.

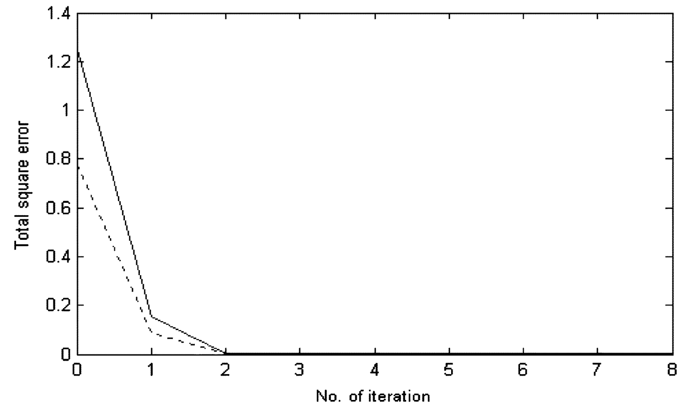

Fig. 8. (Example 1) It shows the total squared error at the interval $t \in [0, 1]$ at different numbers of iterations using ILC rule (15) with estimated system parameters. The solid line represents the total squared error of ILC system output $y_1(t)$ to $y_{r1}(t)$. The dotted line represents the total squared error of ILC system output $y_2(t)$ to $y_{r2}(t)$.

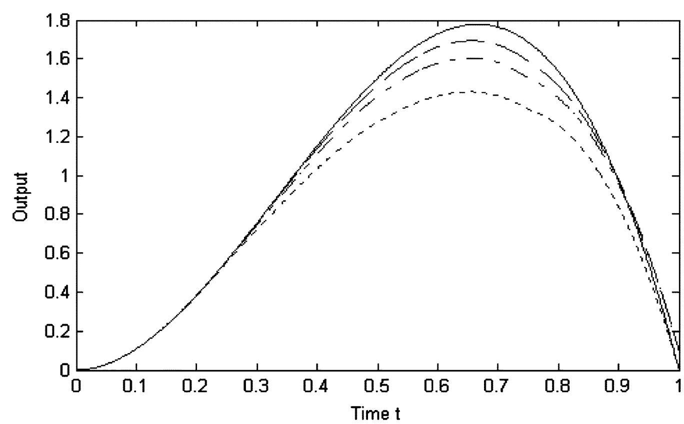

Fig. 9. (Example 2) It shows the outputs of the ILC system for the linear time-delay system (39) using ILC rule (30). The dotted, dashed-dotted, and dashed lines represent the ILC system outputs $y_1(t)$ as the ILC rule (30) is iteratively executed 3, 4, 5 times, respectively, and the solid line represents the desired output $y_{r1}(t)$.

then the determined

$$\hat{K}_1 = -(\hat{C}\hat{B})^T[\hat{C}\hat{B}(\hat{C}\hat{B})^T]^{-1}\hat{C}\hat{A}$$
$$\hat{K}_2 = -(\hat{C}\hat{B})^T[\hat{C}\hat{B}(\hat{C}\hat{B})^T]^{-1}\hat{C}\hat{A}_0$$
$$\hat{K}_3 = (\hat{C}\hat{B})^T[\hat{C}\hat{B}(\hat{C}\hat{B})^T]^{-1}$$

can result in small matrices $CA+CB\hat{K}_1 \neq 0, CA_0+CB\hat{K}_2 \neq 0$, and $I - CB\hat{K}_3 \neq 0$. Because the eigenvalues of $I - CB\hat{K}_3$ are inside the unite circle, the ILC rule (15) is still convergent. Figs. 6–8 show that it takes few more iterations for the ILC rule (15) to drive the tracking error to a very low level for the whole desired output. Our simulation results illustrate that the proposed ILC rules in the form of (8) are robust with respect to small perturbations of the system parameters.

Example 2: In this example, we consider the following linear continuous multivariable system with time delay in input

$$\dot{x}(t) = \begin{bmatrix} -2.5 & 1 \\ 1 & -0.2 \end{bmatrix} x(t) + \begin{bmatrix} 1.2 & -1 \\ 0 & 0.8 \end{bmatrix} u(t - 0.25) + \begin{bmatrix} 0.3 & 1 \\ -1.1 & 0.8 \end{bmatrix} u(t) \tag{39a}$$

$$y(t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} x(t) \tag{39b}$$

where $x(t) = [x_1(t) \;\; x_2(t)]^T$ and $y(t) = [y_1(t) \;\; y_2(t)]^T$. The initial value of state variable is $x(0) = [0 \;\; 0]^T$, and the initial value of input is $u(t) = [0.5 \;\; 0.5]^T$ for $t \in [-0.25, 0]$. The desired output $y_r(t)$ is described as

$$y_r(t) = \begin{bmatrix} y_{r1}(t) \\ y_{r2}(t) \end{bmatrix} = \begin{bmatrix} 12t^2(1-t) \\ \sin(\pi t) \end{bmatrix}, \qquad t \in [0, 1].$$

In the ILC process of system (39), $x(0, k) = [0 \;\; 0]^T$ and $u(t, 0) = [0 \;\; 0]^T$ for $t \in [0, 1]$ and $k = 0, 1, 2, \ldots$ are assumed. Similar to example 1, we firstly apply the proposed ILC rule (30) to the linear continuous system (39) with time delay in input, and set $K = [\begin{smallmatrix} 0.48 & -0.6 \\ 0.65 & 0.18 \end{smallmatrix}]$, which can stabilize the matrix $I - CBK$. Figs. 9 and 10 show the tracking performance of the ILC system output at the interval $t \in [0, 1]$ when the ILC rule (30) is iteratively executed at different times. Also, Fig. 11 shows curves of the total squared error of tracking under the iterative execution of the ILC rule (30).

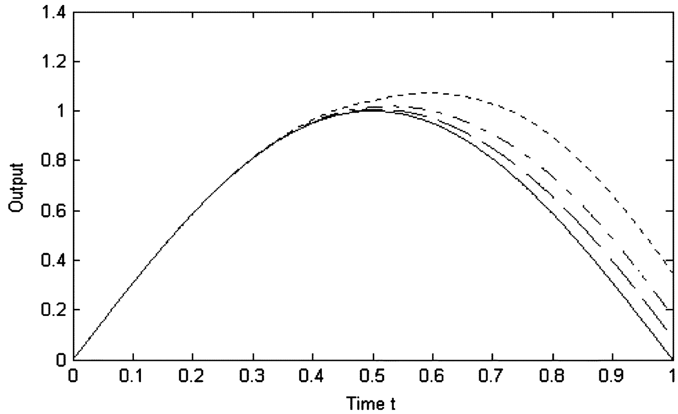

Fig. 10. (Example 2) Outputs of the ILC system for the linear time-delay system (39) using ILC rule (30). The dotted, dashed-dotted, and dashed lines represent the ILC system outputs $y_2(t)$ as the ILC rule (30) is iteratively executed 4, 5, 6 times, respectively, and the solid line represents the desired output $y_{r2}(t)$.

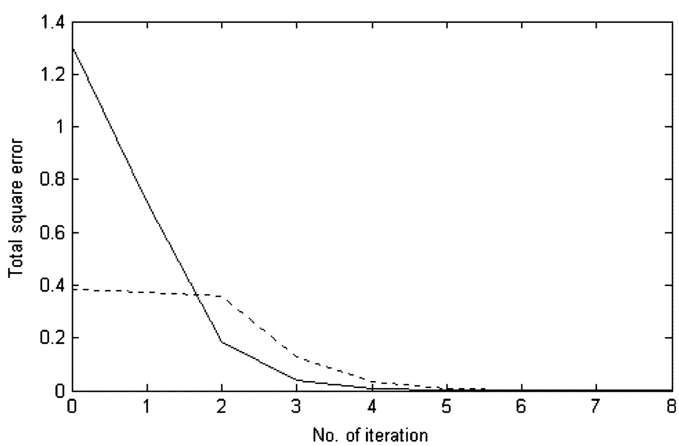

Fig. 11. (Example 2) The total squared error at the interval $t \in [0, 1]$ at different numbers of iteration using the proposed ILC rule (30). The solid line represents the curve of total square error of ILC system output $y_1(t)$ to $y_{r1}(t)$, and the dotted line represents the curve of total square error of ILC system output $y_2(t)$ to $y_{r2}(t)$.

When the ILC rule (31) with $K_1 = -(CB)^T [CB(CB)^T]^{-1}CA$ and $K_2 = (CB)^T[CB(CB)^T]^{-1}$ is used in the same system, Table I shows the total squared error of ILC tracking performance at the different time intervals $t \in [0, 1]$ when ILC rule (31) is iteratively executed different times. It can be noticed that the tracking error is driven to zero for the desired output at the interval $t \in [0, 1]$ only after four times of learning iteration. From Table I, Theorem 5 is verified. It is worth noting that if the desired output $y_r(t)$ is to be extended to the time interval $t \in [0, 1.5]$, according to Theorem 5 the ILC rule (31) is able to drive the tracking error to zero for the desired output at the interval $t \in [0, 1.5]$ only after six times of learning iteration.

Example 3: In this example, the following linear continuous multivariable system with multiple time delays is studied:

$$\dot{x}(t) = \begin{bmatrix} -2 & 0.7 & -1 \\ -1 & 0 & 1 \\ 0 & 1 & -0.5 \end{bmatrix} x(t) + \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -0.5 \\ 0.5 & 0 & 1.8 \end{bmatrix} x(t-0.5)$$

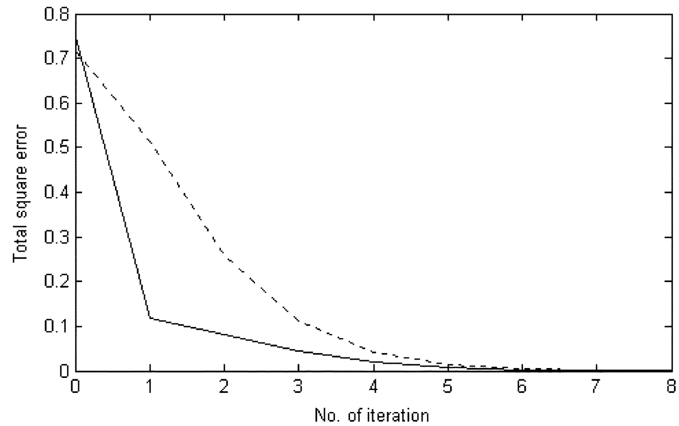

Fig. 12. (Example 3) The total squared error at the interval $t \in [0, 1]$ at different numbers of iterations using the proposed ILC rule (14). The solid line represents the curve of total square error of ILC system output $y_1(t)$ to $y_{r1}(t)$, and the dotted line represents the curve of total squared error of ILC system output $y_2(t)$ to $y_{r2}(t)$.

$$+ \begin{bmatrix} 0.1 & 1 & 0 \\ 0 & 1.2 & -0.3 \\ -1 & 0.2 & 0.5 \end{bmatrix} x(t-0.2) + \begin{bmatrix} 0 & 1 \\ -1 & 0 \\ -0.8 & 1 \end{bmatrix} u(t) \tag{40a}$$

$$y(t) = \begin{bmatrix} 1.2 & 0 & -1 \\ 1 & 0 & 0 \end{bmatrix} x(t) \tag{40b}$$

where $x(t) = [x_1(t)\ x_2(t)\ x_3(t)]^T$ and $y(t) = [y_1(t)\ y_2(t)]^T$. The initial value of state variable is $x(t) = [t\ t\ t]^T$ for $t \in [-0.5, 0]$. The desired output $y_r(t)$ is described as

$$y_r(t) = \begin{bmatrix} y_{r1}(t) \\ y_{r2}(t) \end{bmatrix} = \begin{bmatrix} 12t^2(1-t) \\ 1.5t \end{bmatrix}, \qquad t \in [0, 1].$$

In the ILC process of system (40), $x(t, k) = [t\ t\ t]^T$ for $t \in [-0.5, 0]$ and $k = 0, 1, 2, \ldots$, and $u(t, 0) = [0\ 0]^T$ for $t \in [0, 1]$ are assumed. We apply the proposed ILC rule (14) to the linear continuous system (40) with multiple time delays in state, and set $K = [\begin{smallmatrix} 1 & -0.2 \\ 0 & 0.8 \end{smallmatrix}]$, which can stabilize the matrix $I - CBK$. Fig. 12 shows the curves of the total squared error of tracking two different outputs $y_{r1}(t), y_{r2}(t)$ in the process of ILC rule (14) being iteratively executed. The convergence of ILC rule (14) for linear continuous system with multiple time delays in state is hence verified.

The presented results illustrate that the proposed ILC approaches for linear continuous time-delay systems are very effective. They are able to track the desired output trajectory for the whole time interval after less iteration. It is clear that the ILC rules (14) and (30) are simple. Despite the relatively computational complexity of ILC rules (15) and (31), they are more effective in terms of its tracking error convergence.

V. CONCLUSION

In an ILC process, the main technical difficulty is usually caused by the interaction between the system dynamics and the iterative learning process. Linear continuous multivariable systems with time delays have more complex dynamics than general linear systems. Using the 2-D linear continuous-discrete Roesser's model, this paper describes the ILC process for linear continuous multivariable systems with time delays in state or with time delays in the input. As a result, several ILC rules are derived based on the convergent property of 2-D Roesser's model. Necessary and sufficient conditions for convergence of the proposed ILC rules are given in the form of theorems. It is important that the convergence of the proposed ILC rules is robust with respect to small perturbation of the system parameters. Finally, it is worth noting that the computation of the ILC rule (15) and (31) may require knowledge on the internal states of the ILC systems, and additional observers will then be employed as a result.

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